The spin-glass cornucopia

Marc Mézard

Ecole normale supérieure
PSL University

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40 years of research

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- 1978-…: Theory develops along two lines: mean field theory (Sherrington-Kirkpatrick, Parisi, Mézard Parisi Virasoro…) and « real space » droplets (Fisher- Huse)
- No application, no grant, no nice pictures, but… intellectual curiosity.

Where do we stand?

Real spin glasses: many open questions

A very sophisticated and powerful corpus of conceptual and methodological approaches has been developed in the mean-field approach, and used in many different fields

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Starting in 1984 (assignment, travelling salesman, MM-Parisi), expanding continuously since then.

« *Spin glass as cornucopia* »,
P.W. Anderson, Physics Today 1989

The cornucopia

- Physics (all kinds of glassy phases)
- Computer Science (constraint satisfaction problems: satisfiability, coloring, TSP,…)
- Information Theory (error correcting codes)
    - Signal acquisition and processing - big data
    - Significant information
    - Gene expression networks
    - Neural networks
    - Interacting agents on a market

Why?

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Last forty years : handle strongly disordered systems. Each atom has a different environment. Even the nature of the basic phases is hard to understand.
Heterogeneous « agents »

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Beginning of statistical physics and condensed matter

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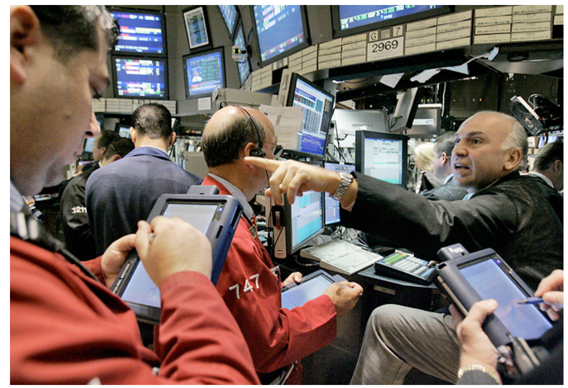


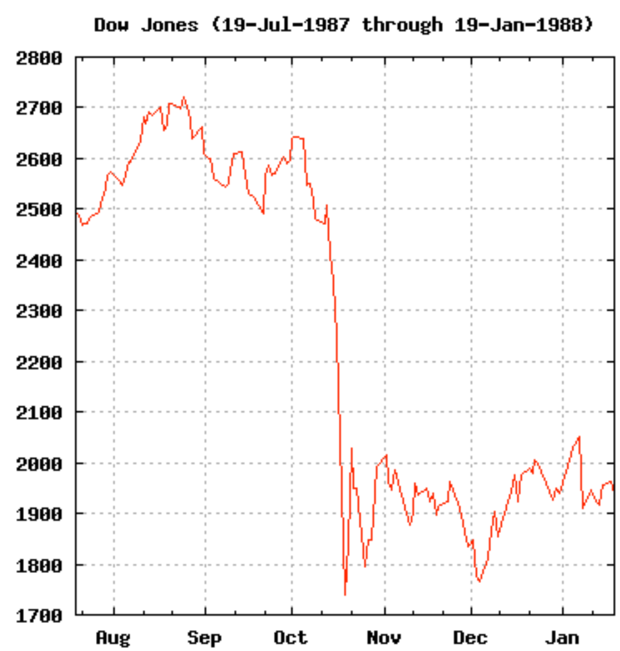
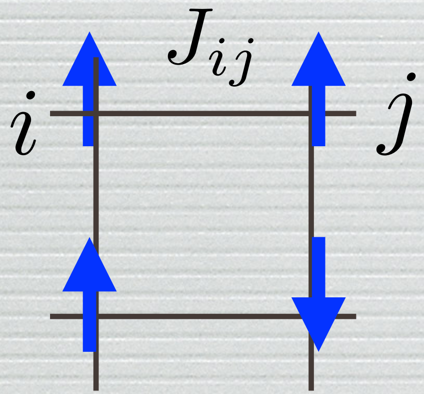


Table of content

- **Spin glasses** and structural glasses : slow dynamics, mean field description
- Information theory: phase transitions and glass phases in **error correcting codes**
- **Computer science:** constraint satisfaction problems
- Cavity method, mean-field equations and algorithms

Magnets



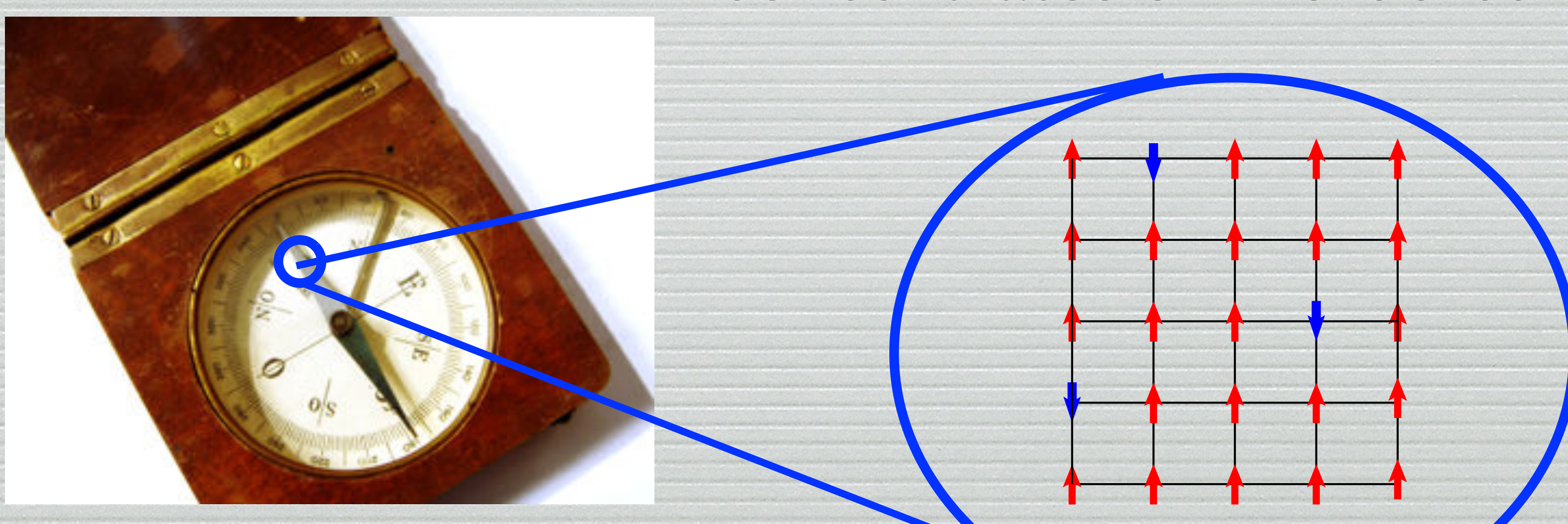
$$s_i \in \{\pm 1\}$$

$$E = -\sum_{ij} J_{ij} s_i s_j$$

Equilibrium: $P(s_1, \ldots, s_N) = \frac{1}{Z} e^{-E/T}$

Ferromagnet: $J_{ij} > 0$

At low T: spins align, P concentrates on 2 ordered states



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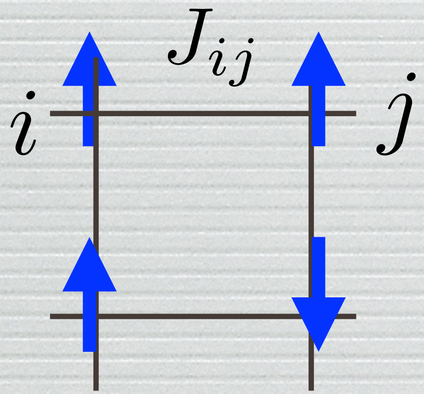

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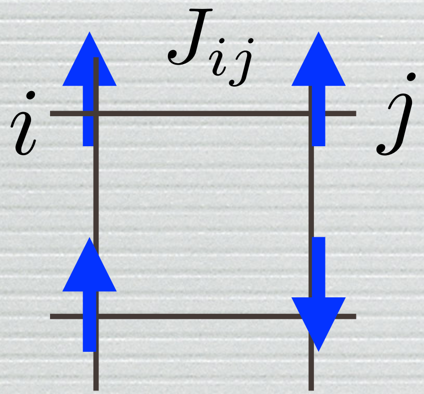

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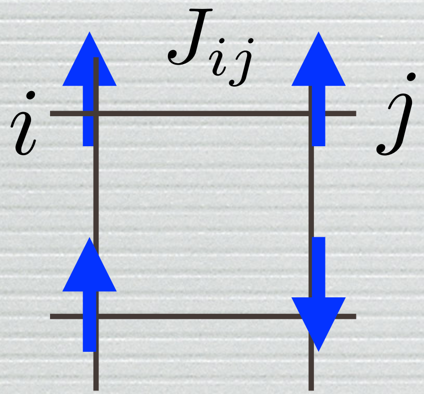

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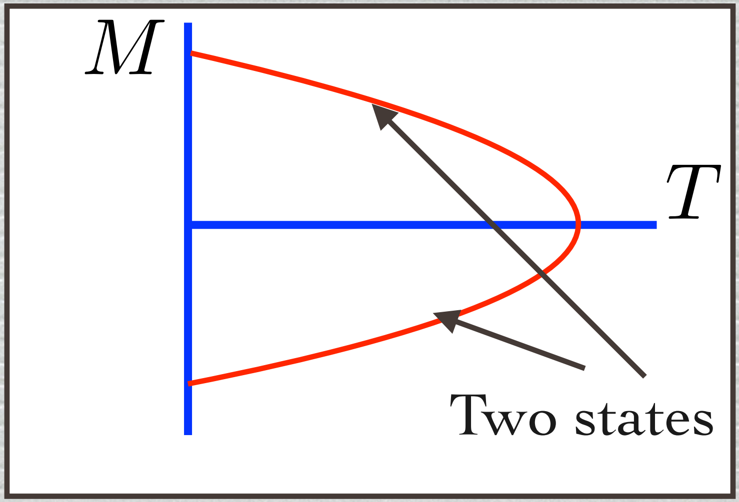

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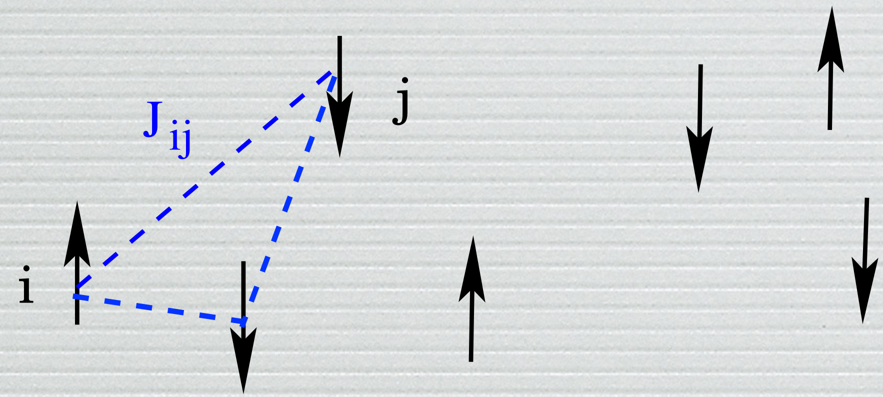
Spin glasses

CuMn



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Spin glass: J_{ij} sign depends on ij $\Longrightarrow$ **frustration**

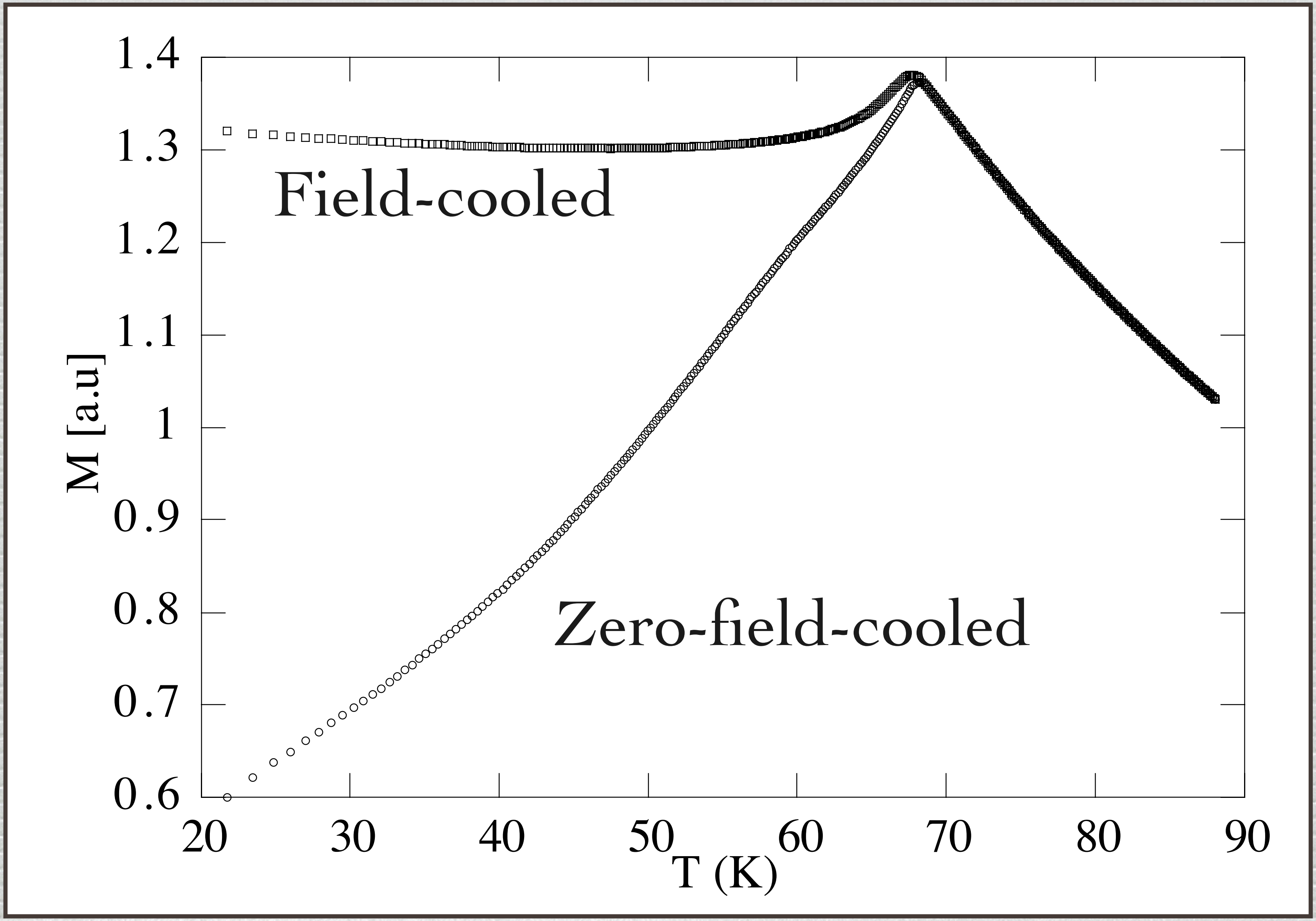
Frustrated triplet: $J_{ij} J_{jk} J_{ki} < 0$

Disorder and frustration are the two building blocks of spin glasses

What is the behavior at low T ?

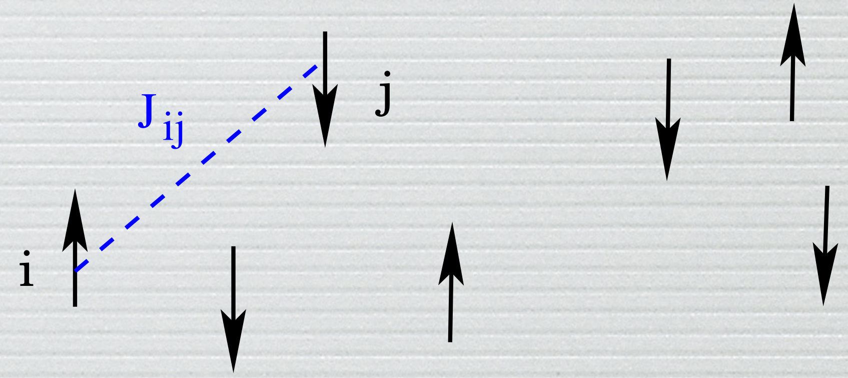
Magnetization, linear response to a small magnetic field

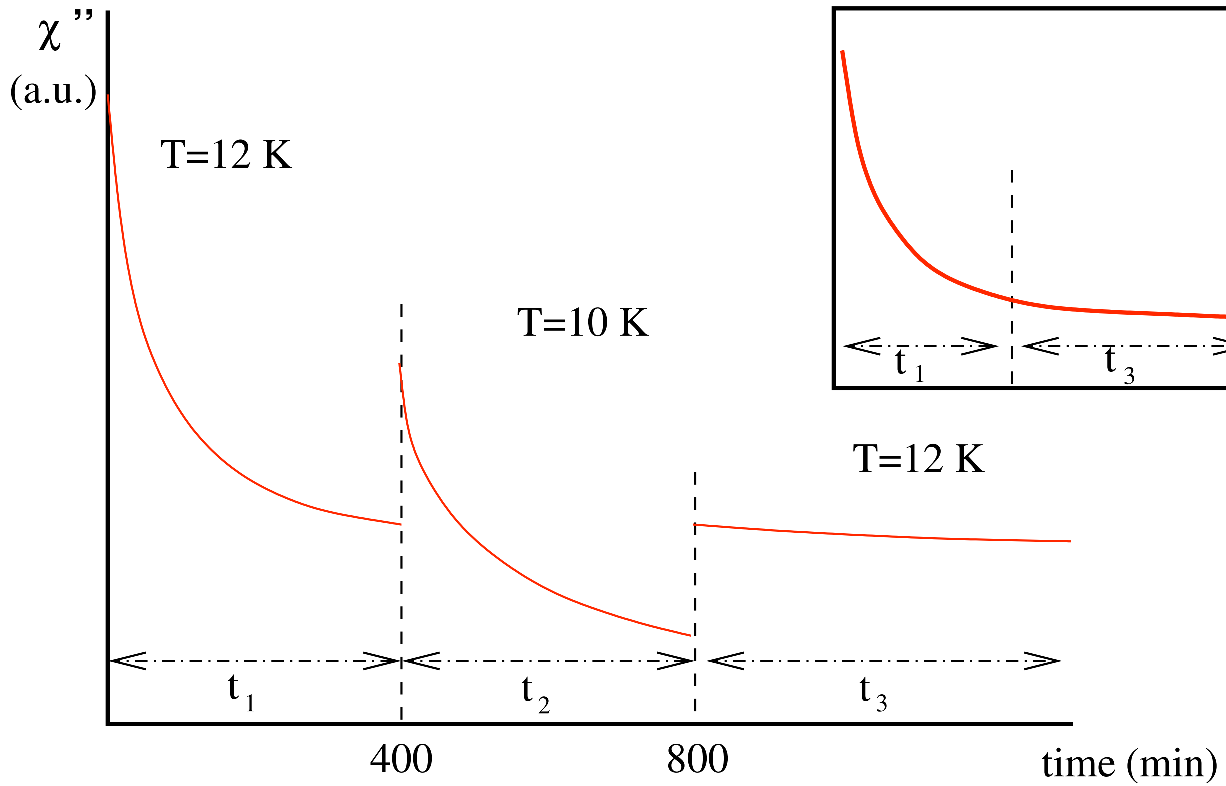
Non-equilibrium.



Spin Glasses

Linear response to a small magnetic field:





New dynamics
Memory

E. Vincent et al, SPEC

Spin glasses

What is the behavior at low T?

out-of-equilibrium effects are crucial

1- How does the equilibrium P behave at low T ?

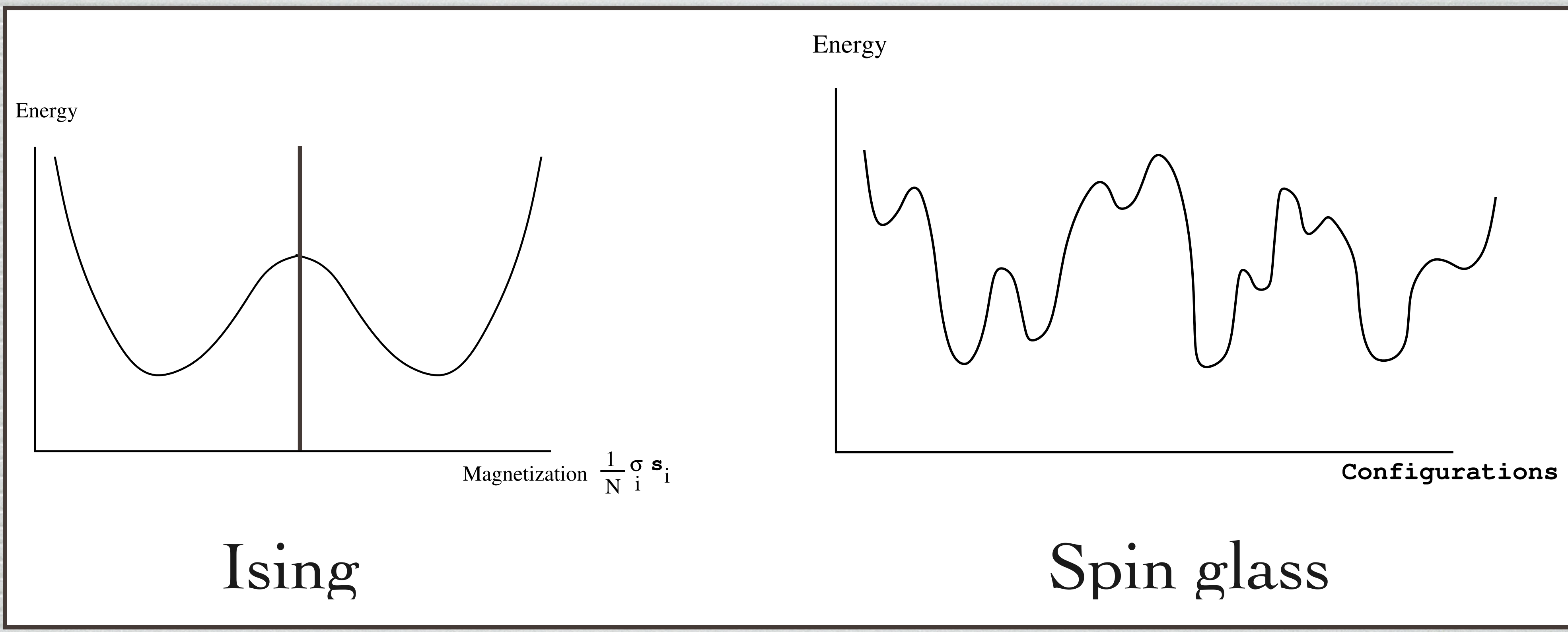
2- Study the dynamics : What causes the very slow dynamics? Quasi-equilibrium?

Mean-field **ensemble**: $$E = -\sum_{ij} J_{ij} s_i s_j$$

SK model

J_{ij} iid Gaussian

Mean-field lessons



1- Glass « phase » : Many pure states, unrelated by symmetry, organized in a hierarchical « ultrametric » structure $T < T_c$

2- Many metastable states, unrelated by symmetry

3- « True » ground state : fragile to perturbation!

Equilibrium: order parameters

Ferromagnet: $$M^{\pm} = \lim_{B \to 0^{\pm}} \langle s_i \rangle_B$$

Spin glass: $$M^{\alpha} = \lim_{B_i \to 0^{\pm(\alpha)}} \langle s_i \rangle_B$$

Freezing into an unknown, disordered state: unwieldy!

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Use the system itself as a conjugate field: **replicas**

Overlap between two equilibrium configurations

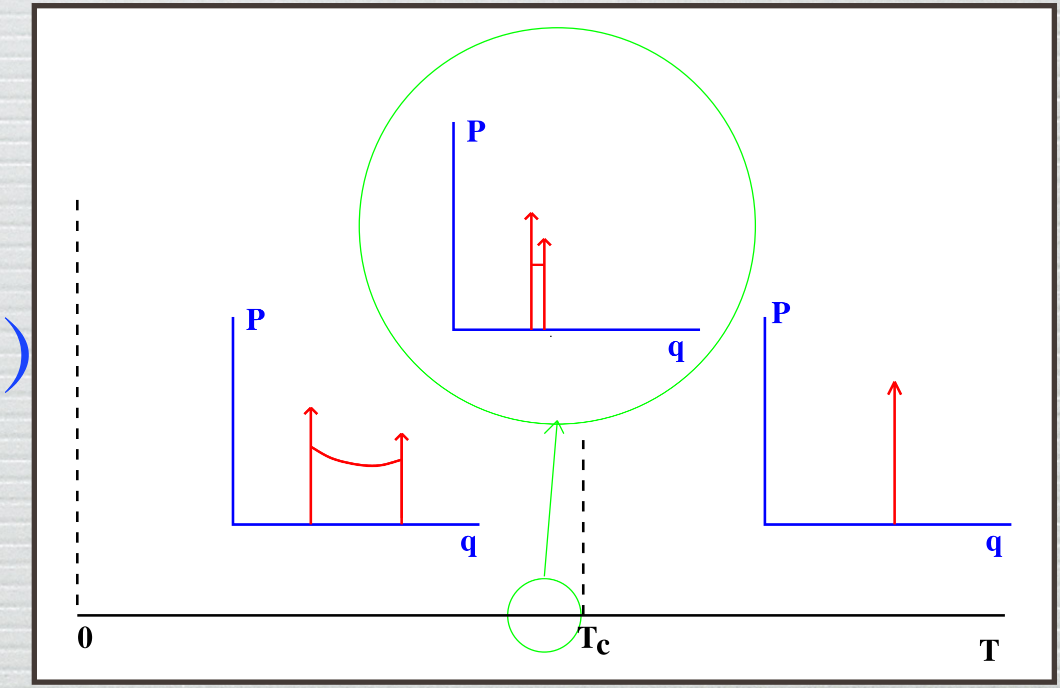
$$q = \frac{1}{N} \sum_i s_i^1 s_i^2$$

Order parameter = Probability of overlap q : $\boxed{P(q)}$

Two families of spin glasses

Two families of spin glasses

Probability (2 random configurations have overlap q)



Continuous transition

« Full replica symmetry breaking »

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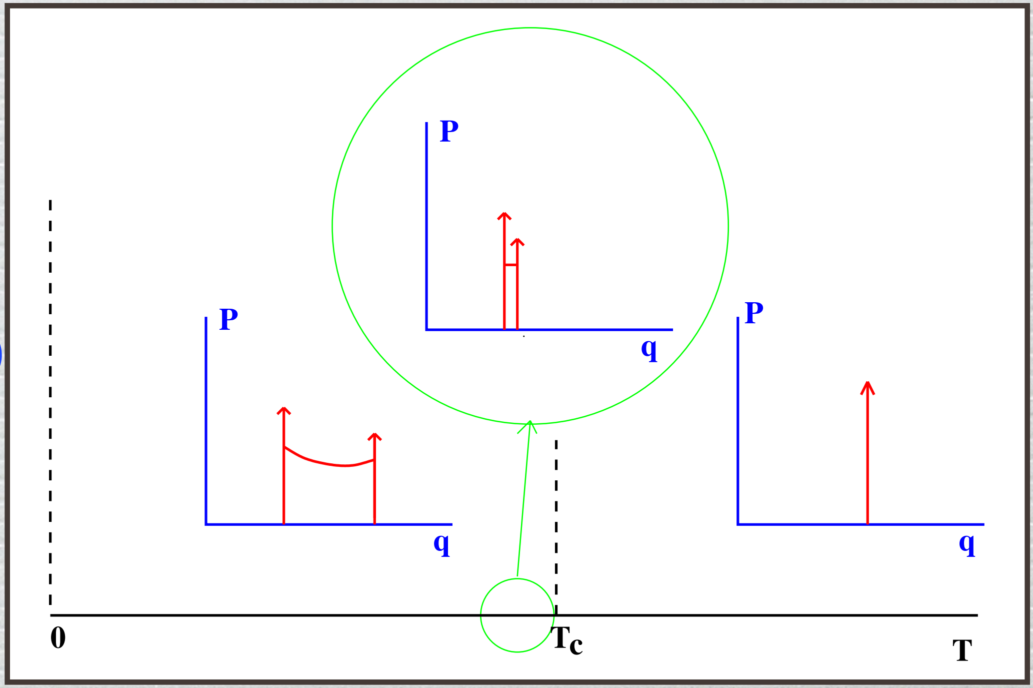

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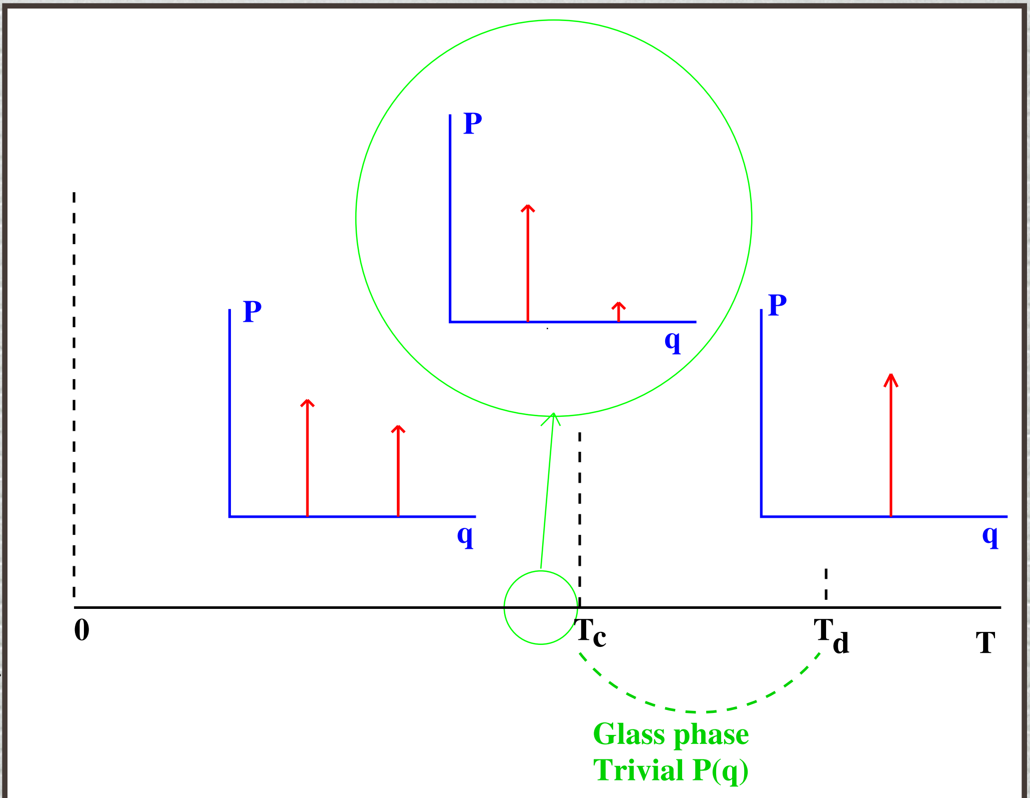
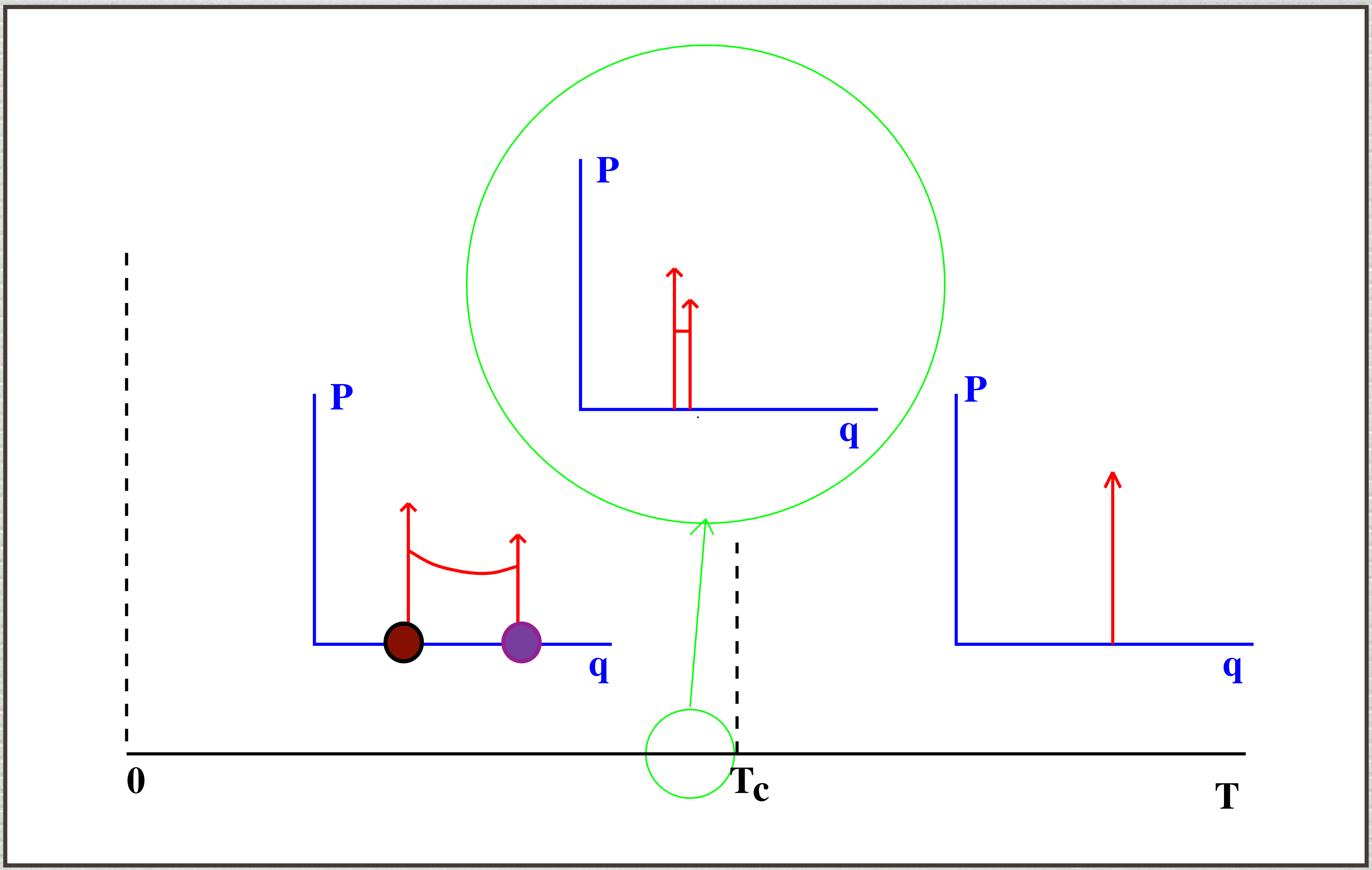


Discontinuous transition

« One step replica symmetry breaking »

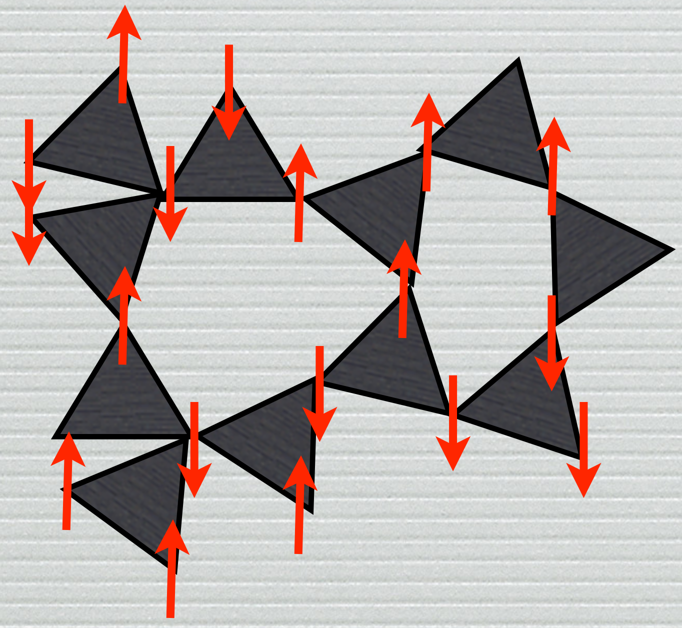


Two replicas with small attraction

Two replicas with small repulsion

Example of a spin glass model with a discontinuous transition

3-spin interaction



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$\downarrow \quad s_i = -1$

$$E = -\sum_{ijk} s_i s_j s_k$$

Randomly chosen triplets

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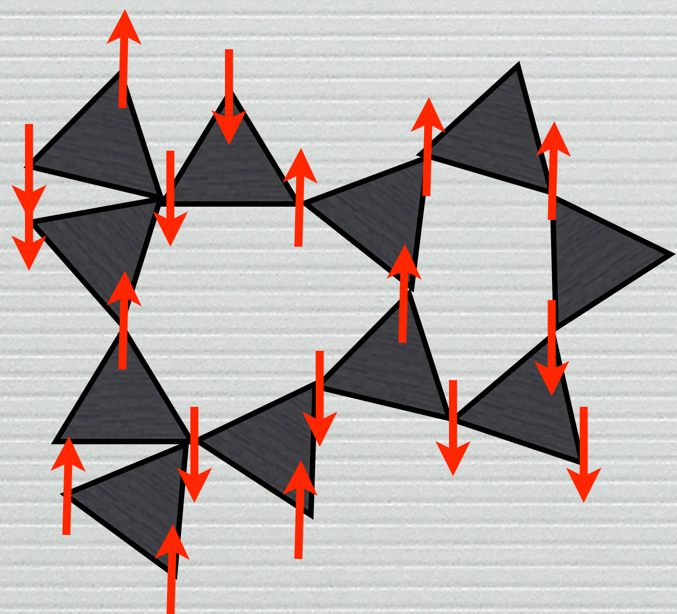

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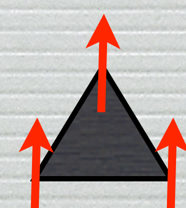
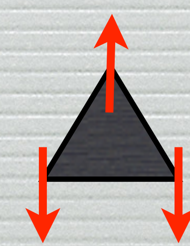
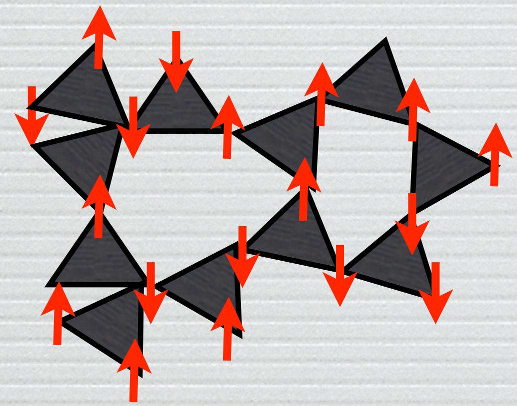
Lowest E:  or  or...

Trapped in a glass phase



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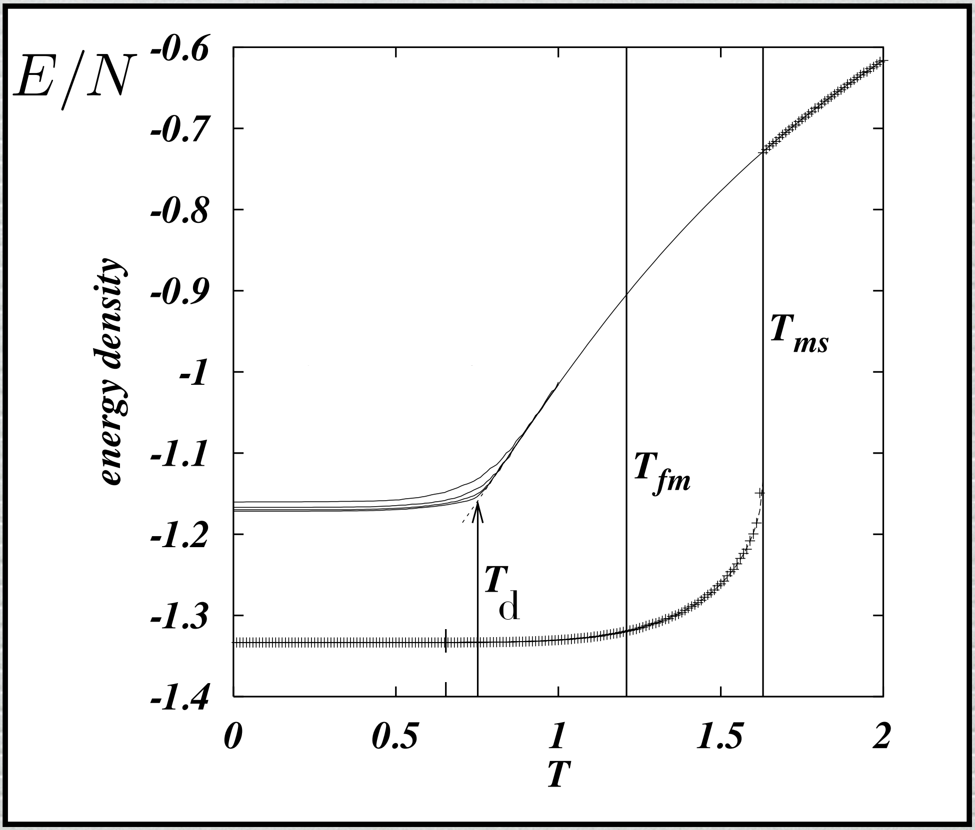
10^5 spins, 4 triangles per spin

E/N



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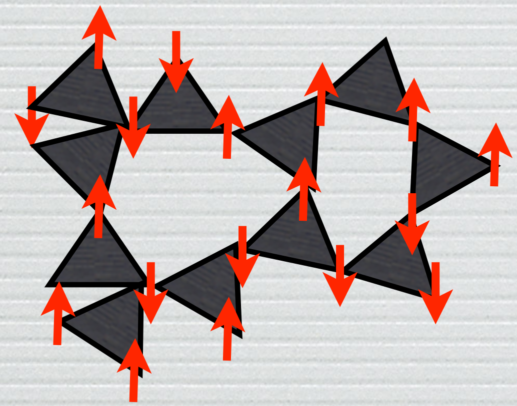

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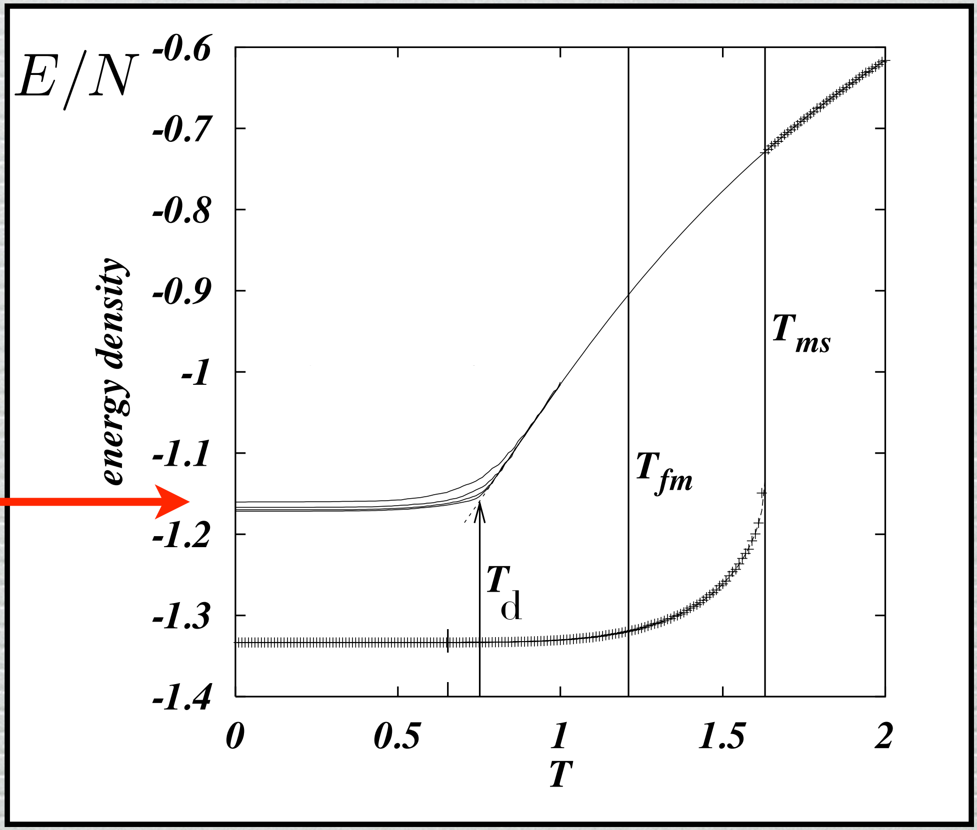
Metastable states found by simulated annealing 10^4 to 10^7 steps : glass



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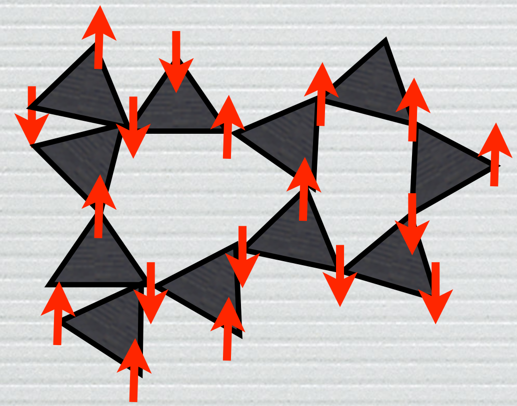

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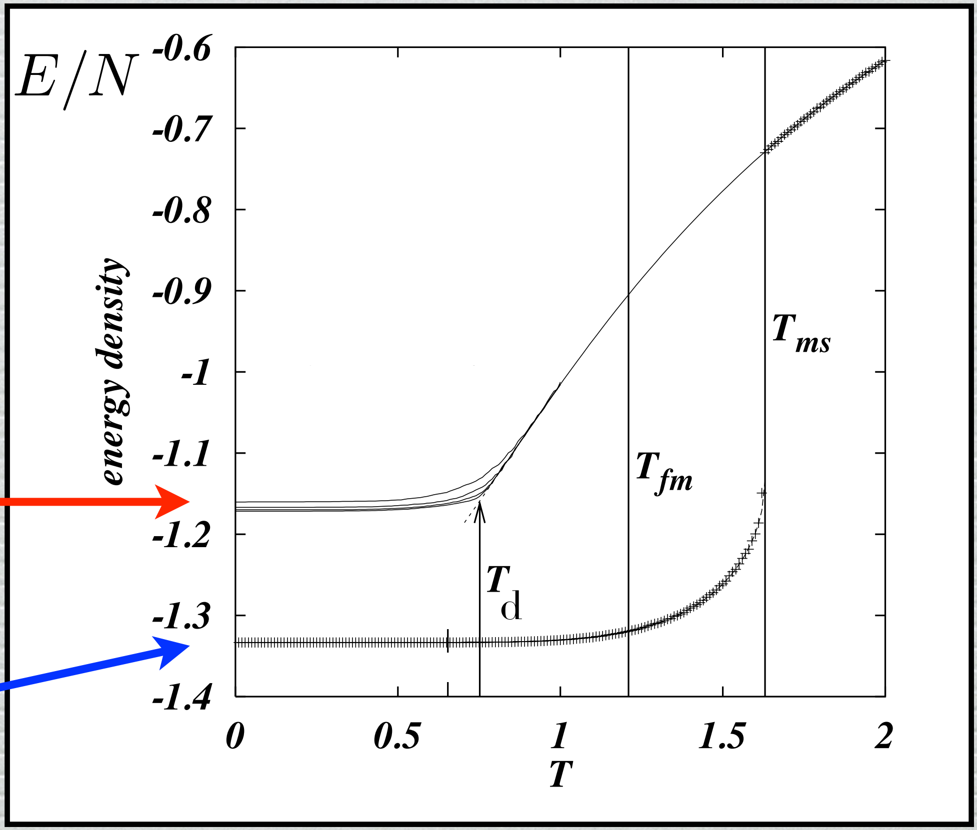
Crystal state, all $s_i = 1$



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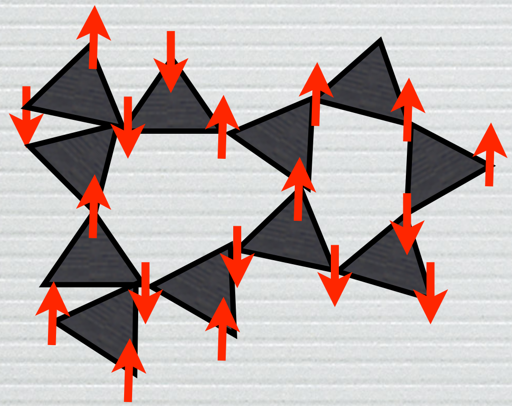

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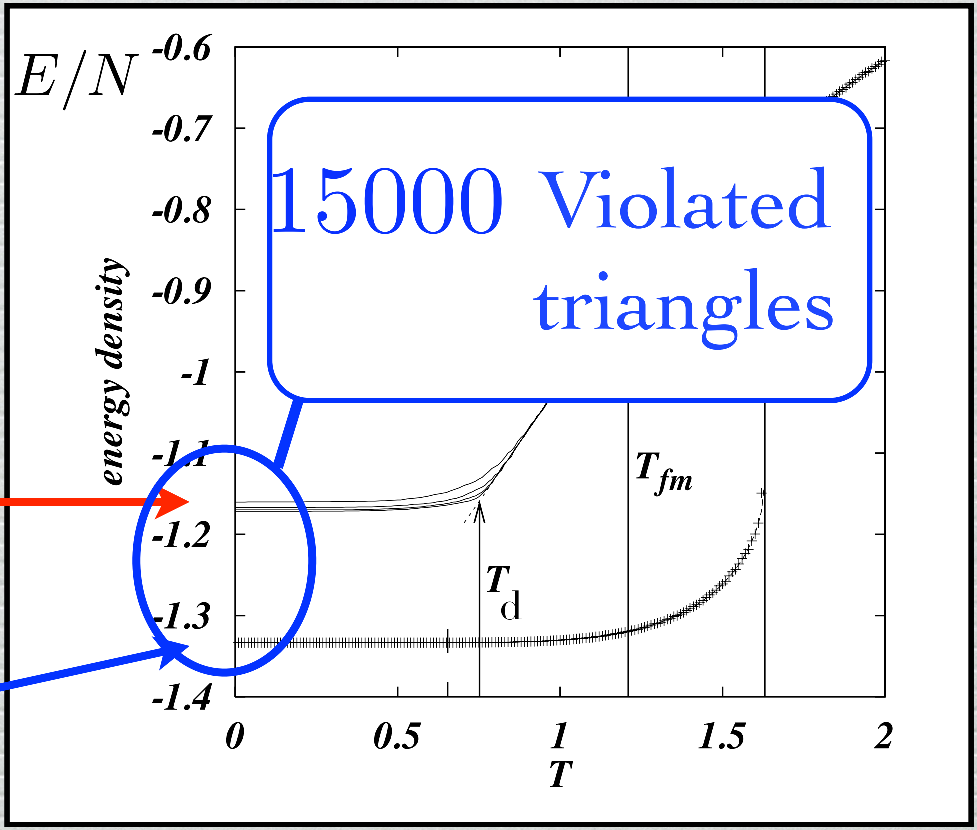

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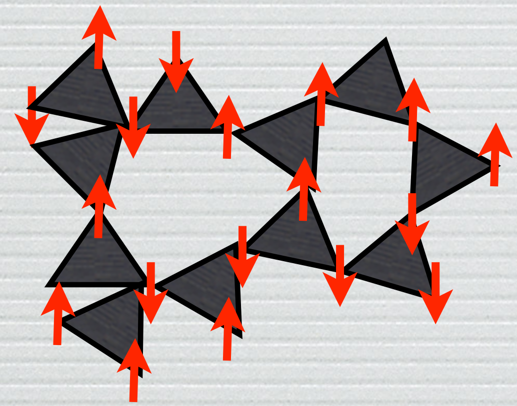

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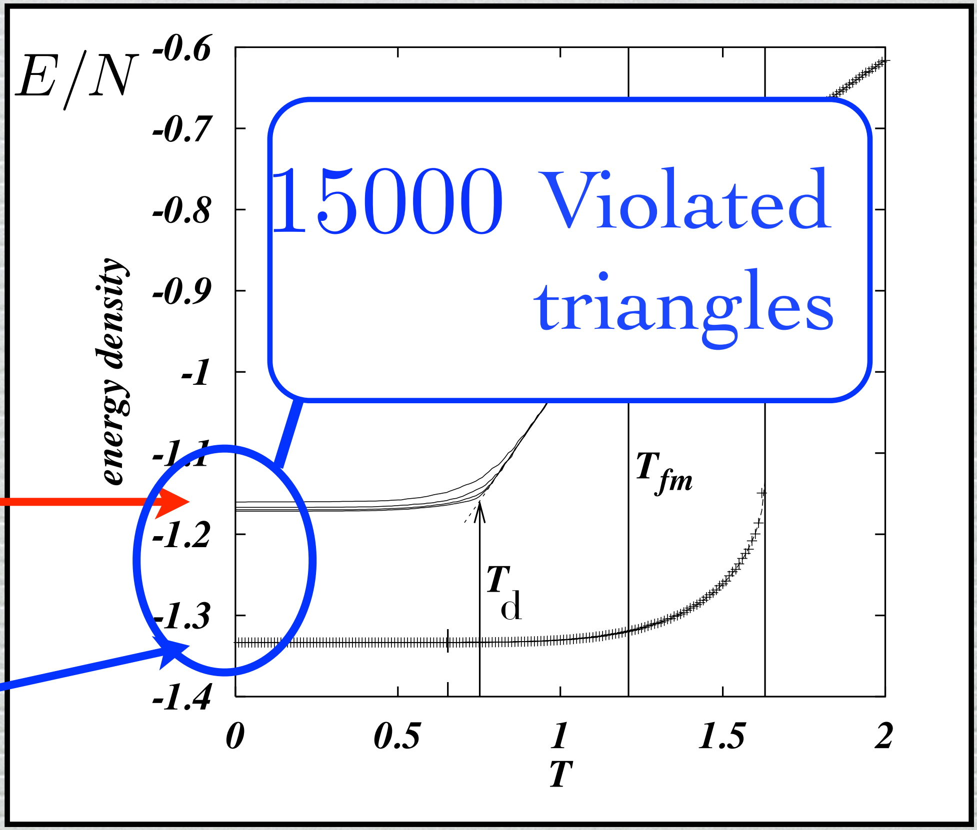
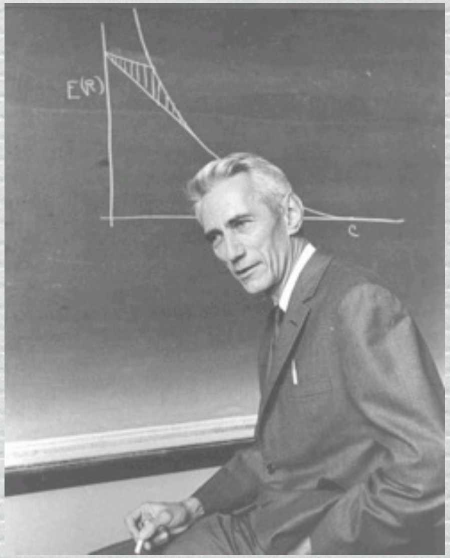


Random first order phase transition at T_c : traps

Collective behaviour and emergent properties of systems made from many **different «atoms»**:

- Magnetic moments (spin glasses)
- Molecules (structural glasses)
- Information bits(information theory, coding)
- Neurons, spikes (neural networks)
- Gene activities (gene expression networks)
- Logical variables (constraint satisfaction)
- Information bits (error correction)
- Agents on a market (finance, game theory)
- ...

An example from Information Theory



Claude Shannon
(1916-2001)

Deeply linked to statistical physics

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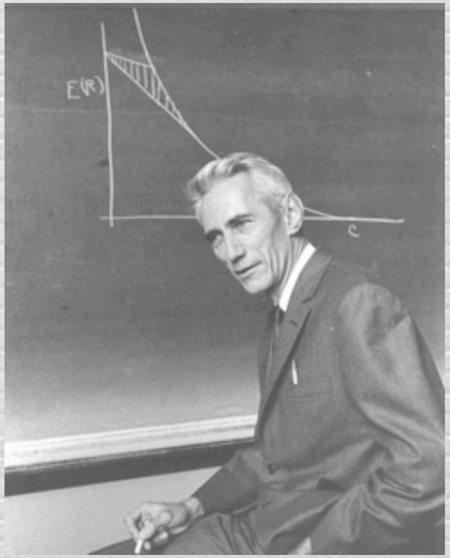

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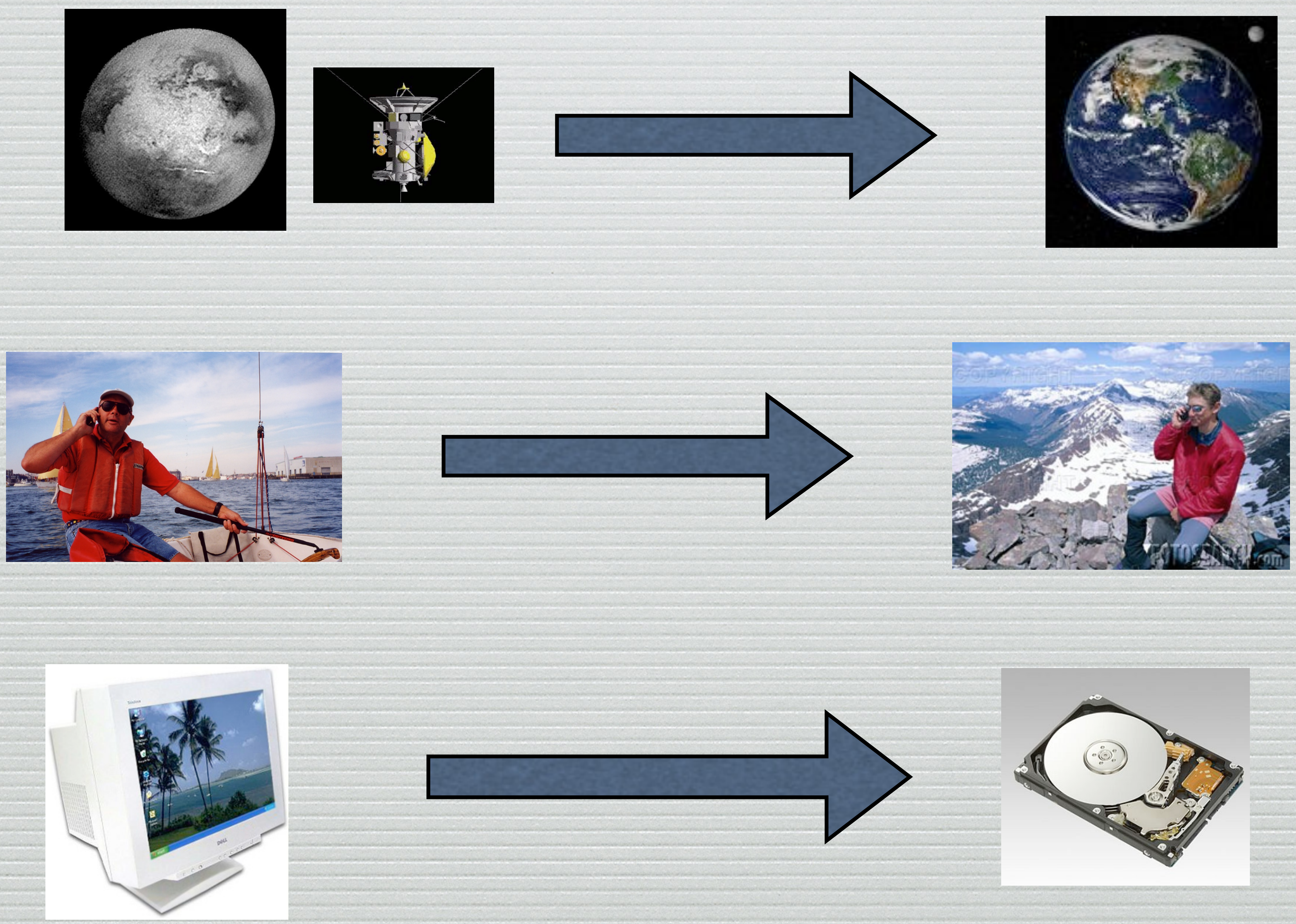
More recent convergence:
Error correcting codes, pool testing, compressed sensing

Information transmission and error correction



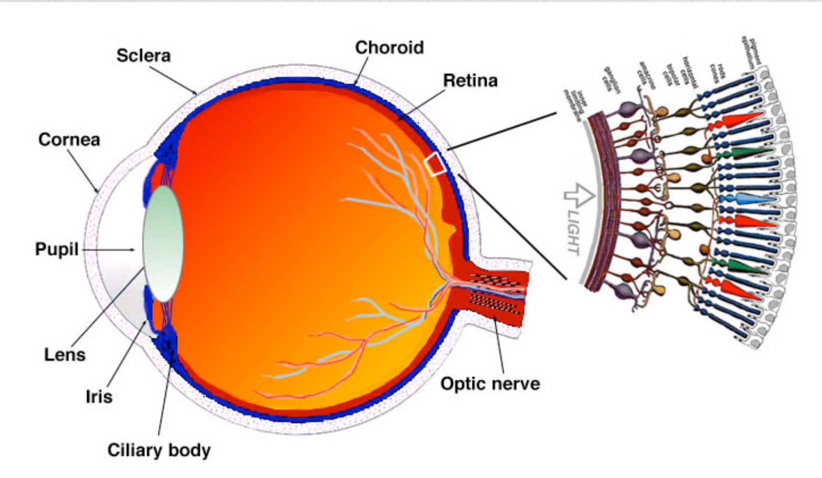
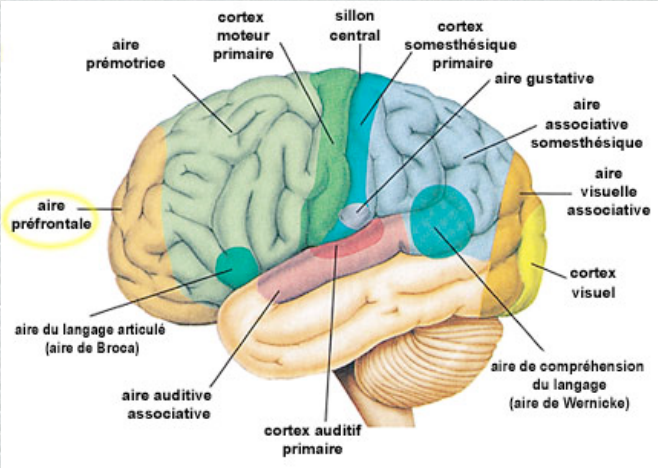
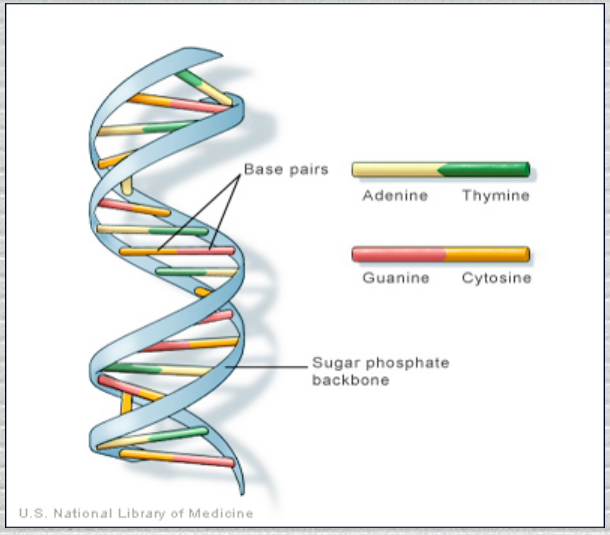
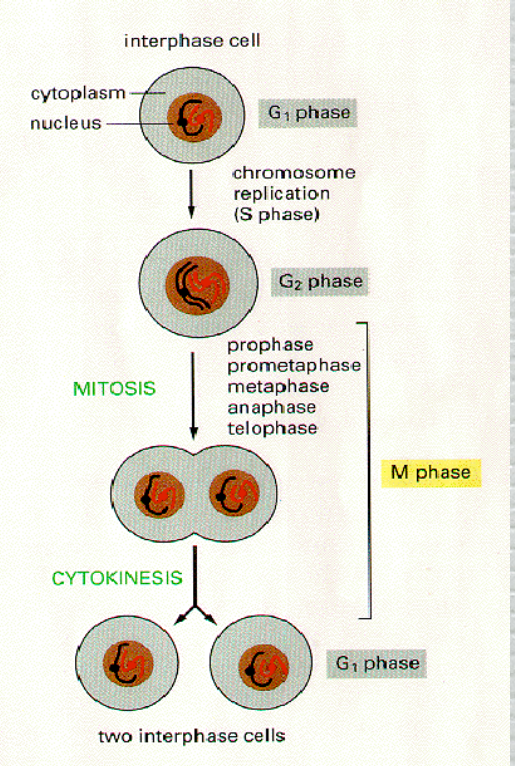


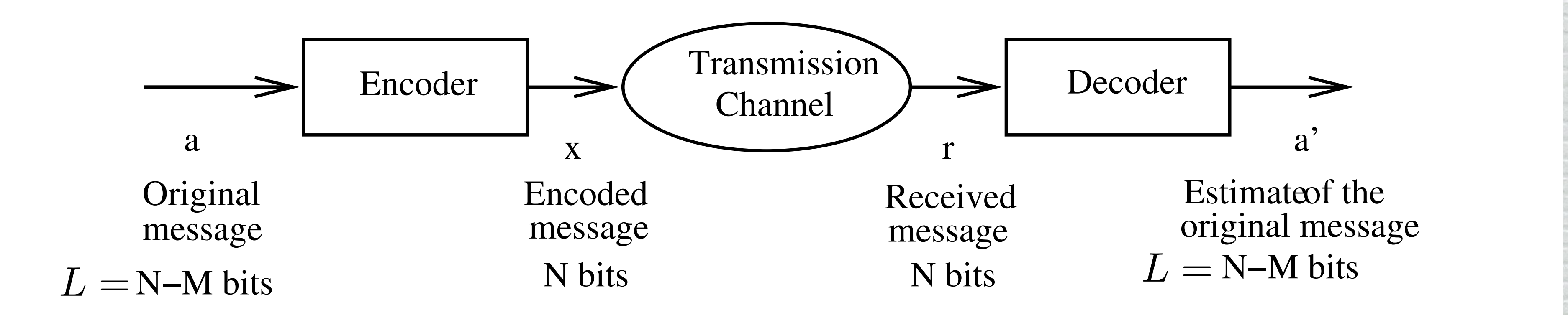
Fig. 1.1. A drawing of a section through the human eye with a schematic enlargement of the retina.







Principle of error correction : redundancy



Encoding = add redundancy. Rate L/N

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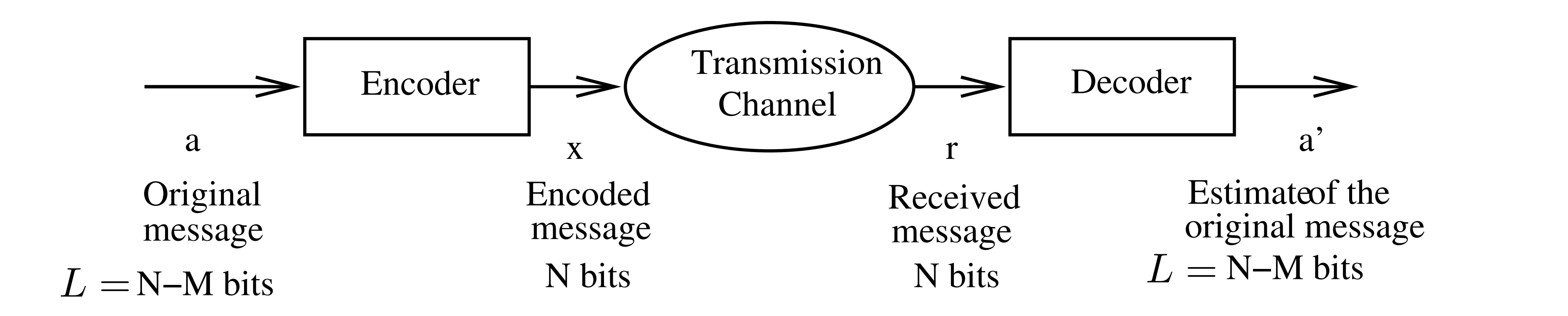

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e.g. repetition $\quad 0 \rightarrow 000 \quad 1 \rightarrow 111 \quad$ rate $= 1/3$

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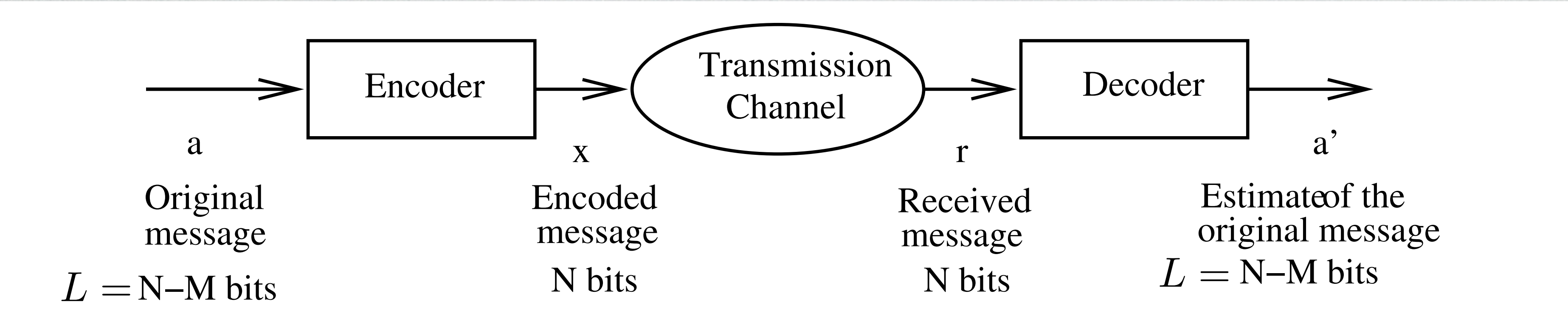

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e.g. repetition $0 \to 000 \quad 1 \to 111 \quad \text{rate} = 1/3$

1011 ⟶ 111000111111 ⟹ 111 000 110 111

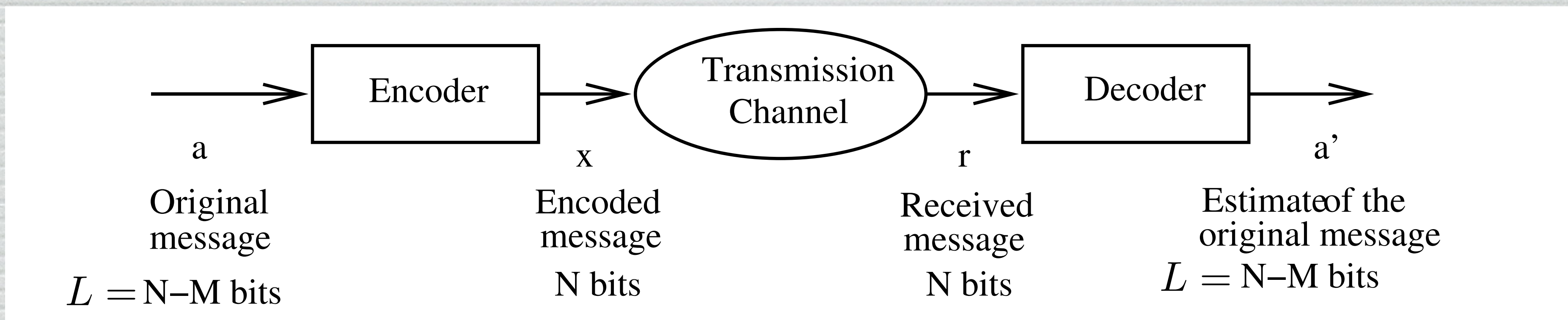
Code Channel

1 0 1 1

Majority decoding

error probability $p^3 + 3p^2(1-p) \sim 3p^2$

Principle of error correction : redundancy



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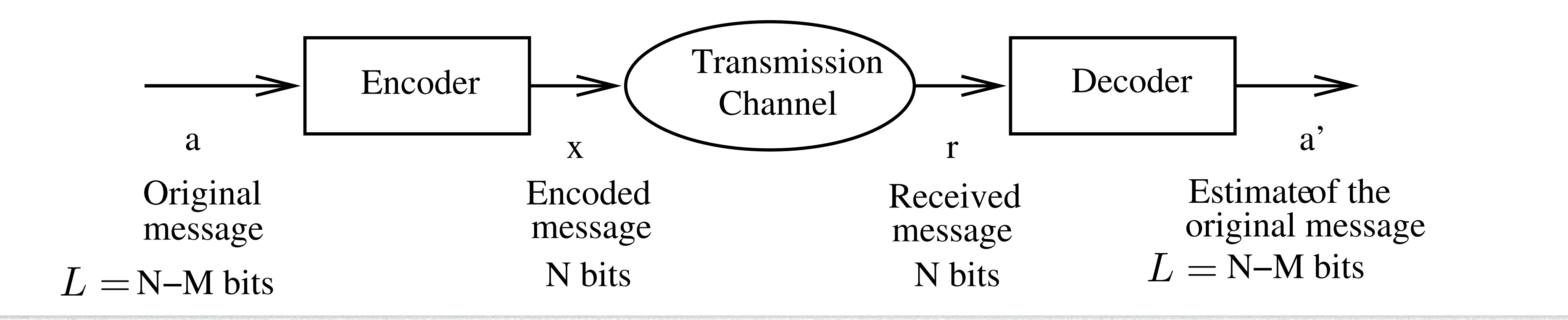
Shannon's theorem: for a given noise level p, one can build a coder/decoder which transmits with **zero error**, iff $r < r_c(p)$

Principle of error correction : redundancy



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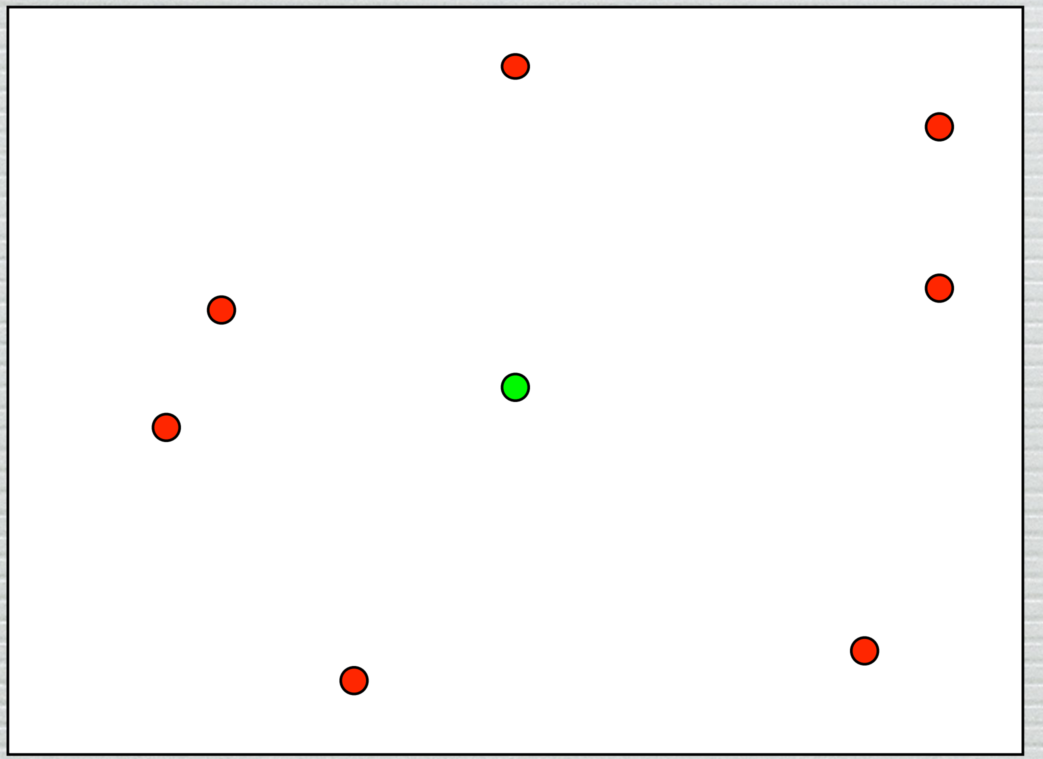
Two ingredients:

- « Thermodynamic limit » $N, L \to \infty$
- Ensemble of Random Codes ($\sim$ Random Energy Model of spin glasses)

Shannon code *ensemble*

Unit hypercube in N dimensions



codewords (random)

sent codeword

Channel: flips bit with probability p

2^{RN} iid random points, uniform distribution

Shannon code *ensemble*

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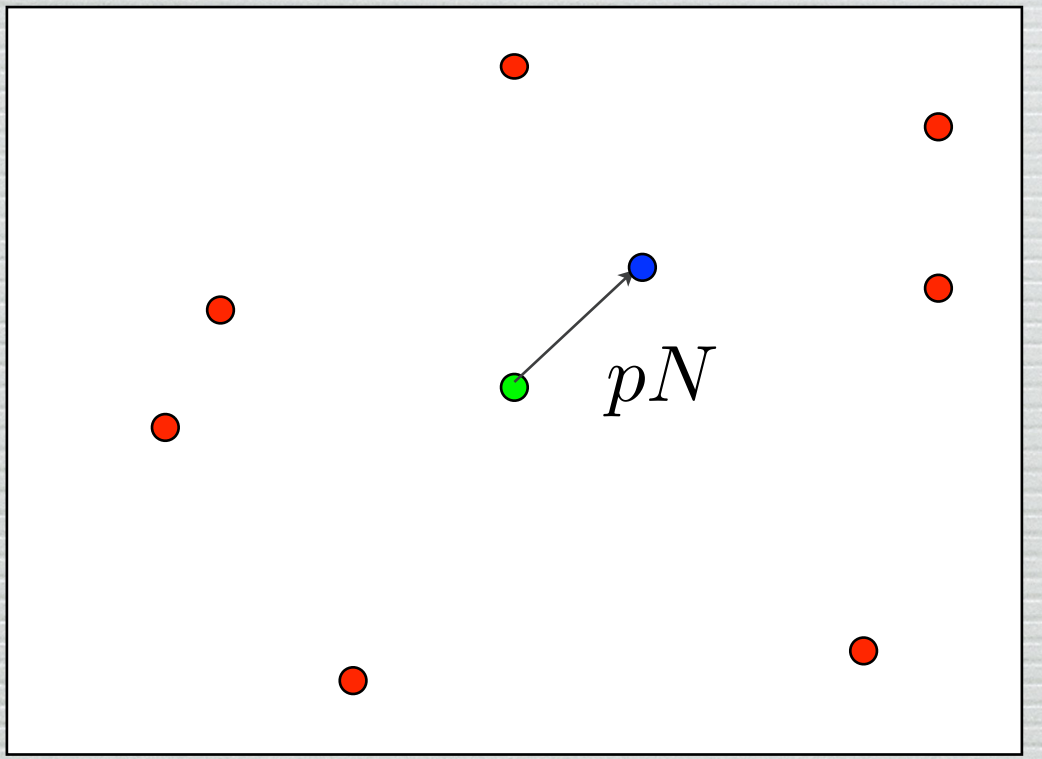

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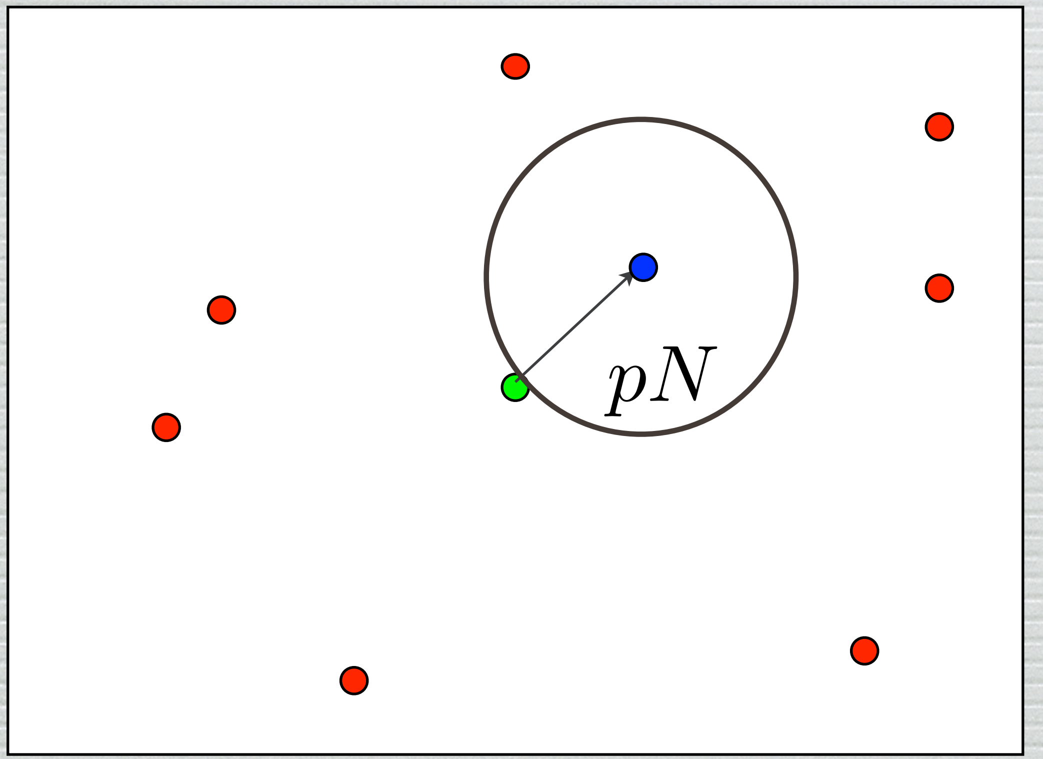
Shannon code *ensemble*

Unit hypercube in N dimensions



codewords (random)

sent codeword

received word

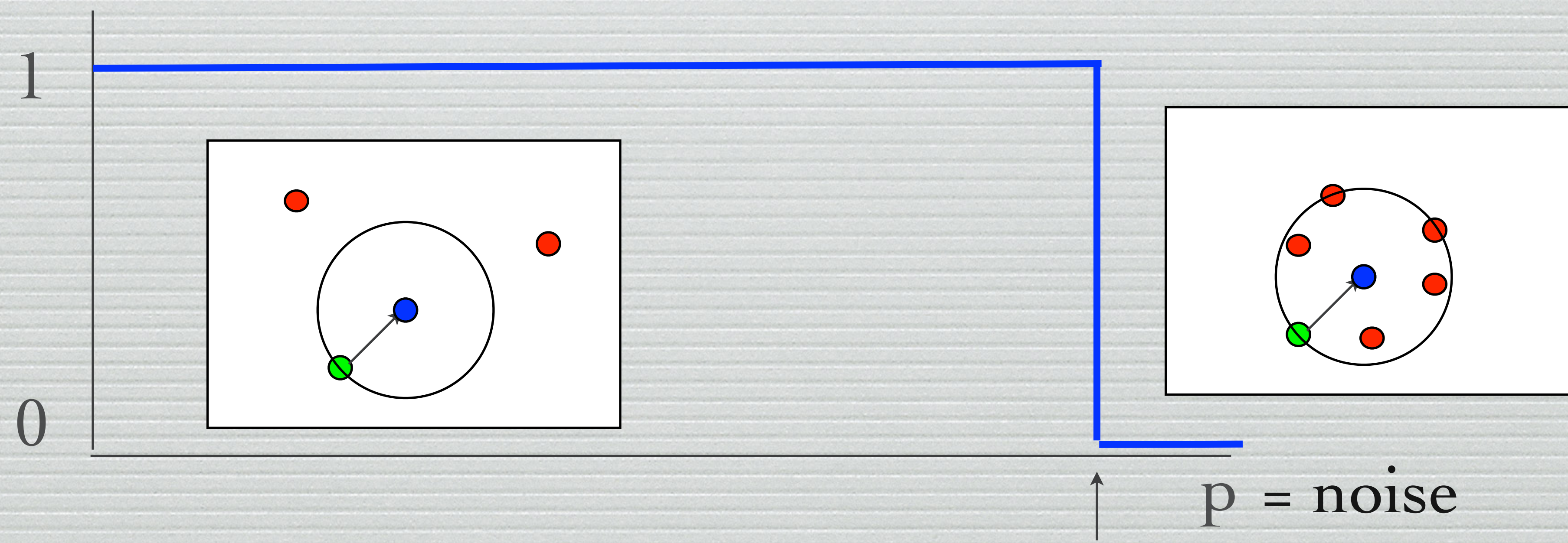
Channel: flips bit with probability p

2^{RN} iid random points, uniform distribution

Phase transitions in decoding

Decoding = find closest codeword

Probability of perfect decoding:

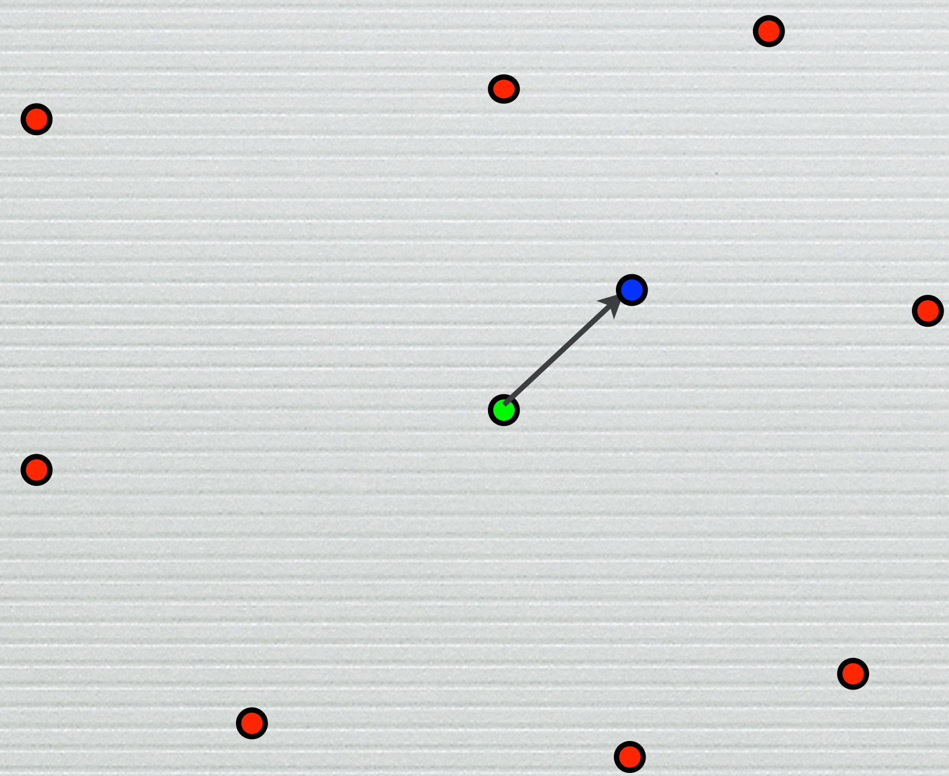


Shannon « bound »

geometric phase transition

Shannon code ensemble

Unit hypercube
in N dimensions



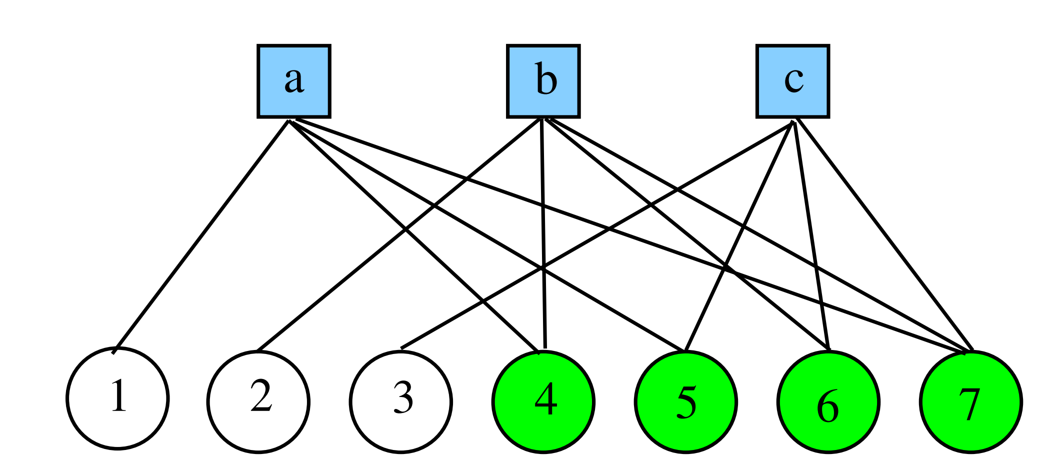
- codewords (random)
- sent codeword
- received word

Perfect codes in principle, but impossible to decode in practice (structureless $\longrightarrow$ time $O(e^N)$)

Efficient codes : parity checks

Shannon's code is useless (time $O(e^N)$)

Add redundancy, with structure allowing to decode



$$x_1 + x_4 + x_5 + x_7 = 0 \pmod 2$$
$$x_2 + x_4 + x_6 + x_7 = 0 \pmod 2$$
$$x_3 + x_5 + x_6 + x_7 = 0 \pmod 2$$

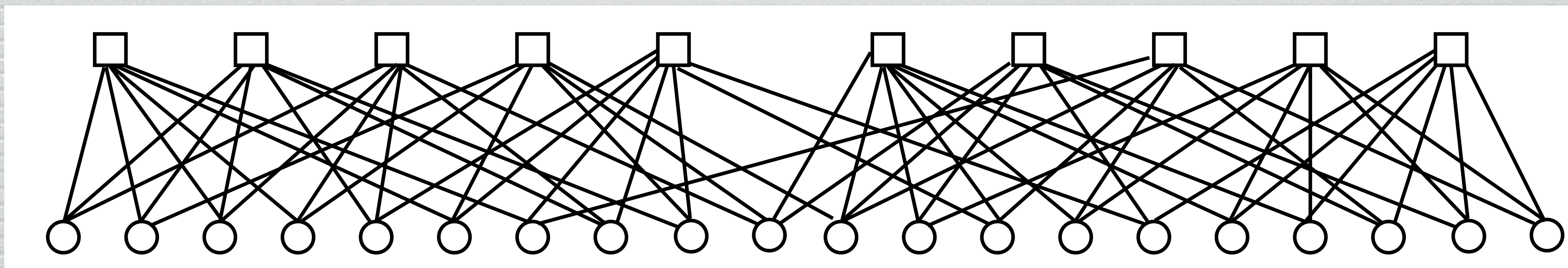
2^4 codewords

among 2^7 words

Error decoding with parity checks

$N \gg 1$ variables, $M = (1-R)N$ equations

Random construction with K variables per equation, L equations per variable

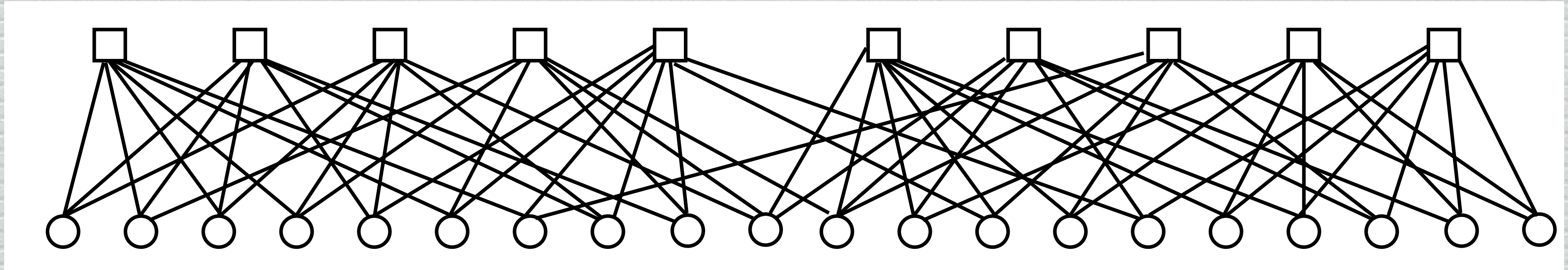


$N = 20, M = 10, R = 1/2, L = 3, K = 6$

Error decoding with parity checks

$N \gg 1$ variables, $M = (1-R)N$ equations

Random construction with K variables per equation, L equations per variable



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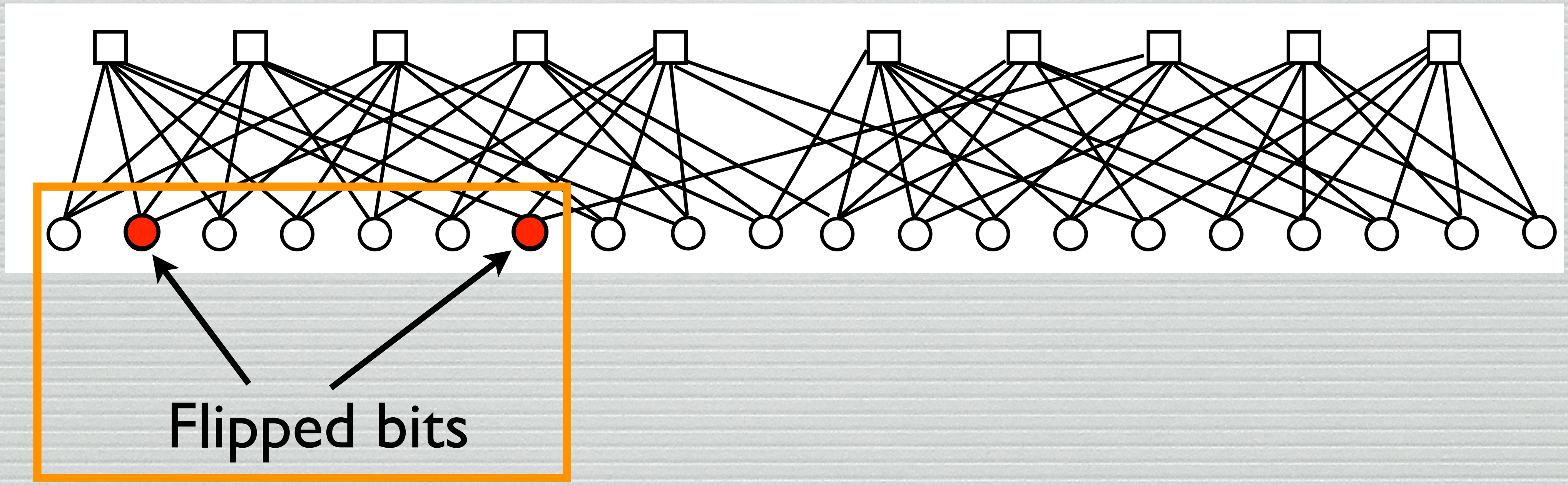

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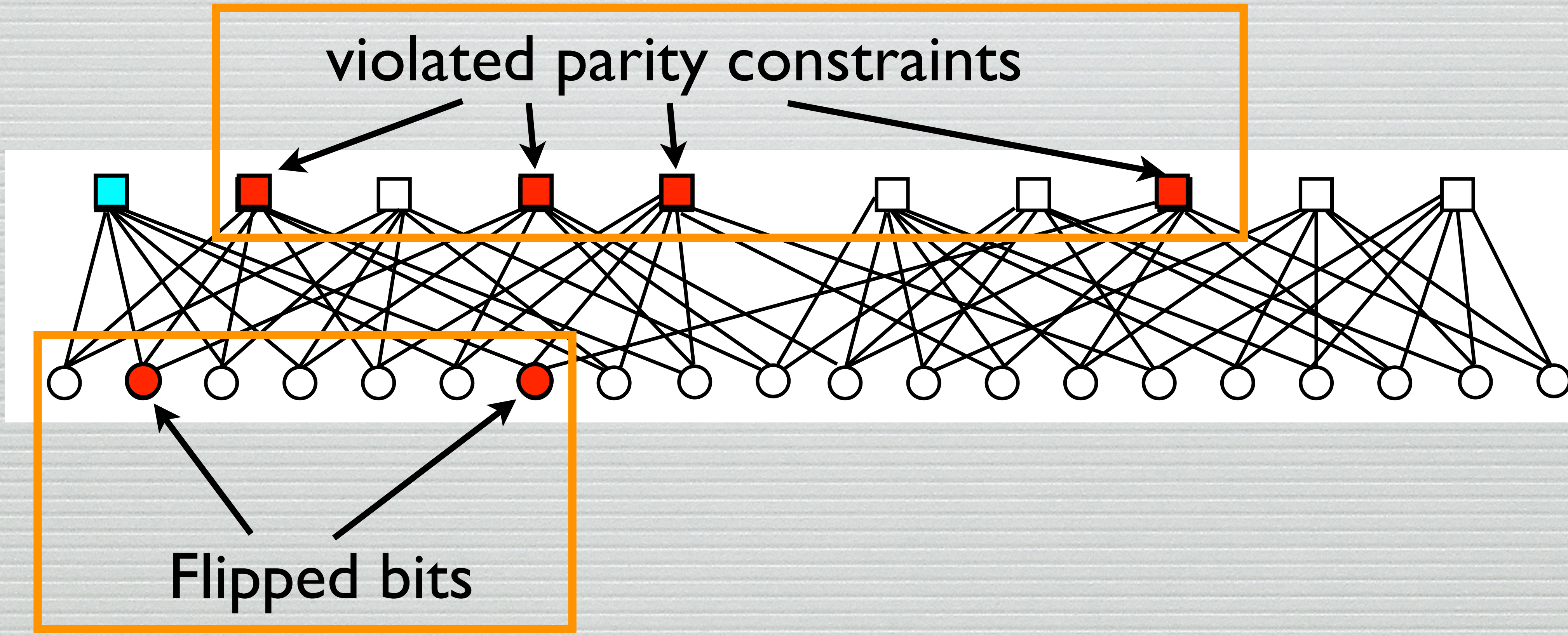

Error decoding with parity checks

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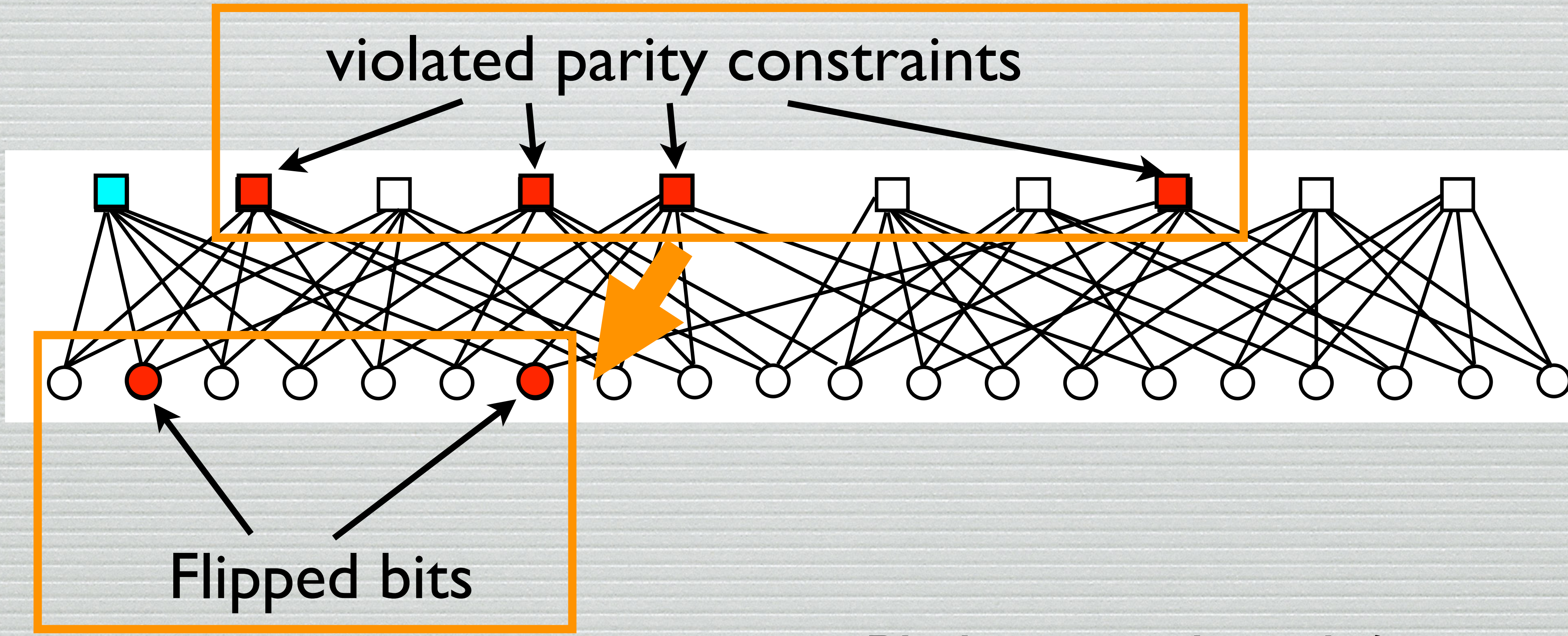
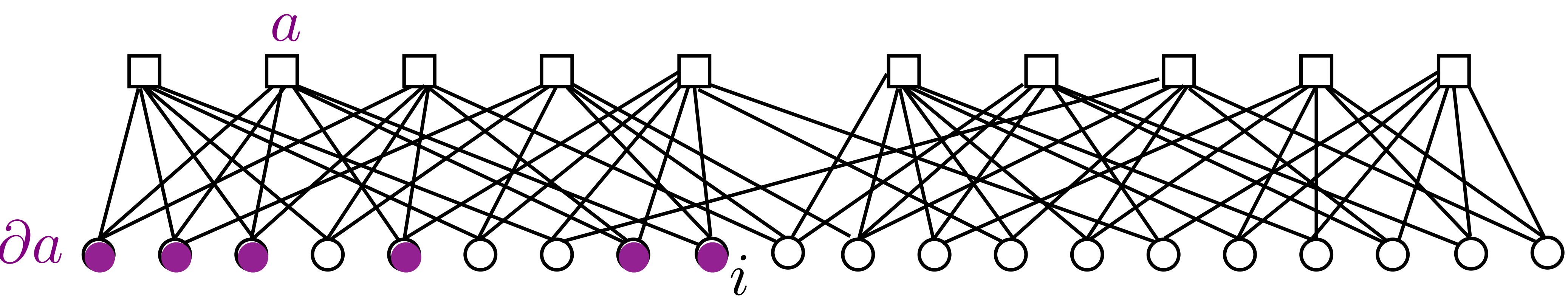
Random construction with K variables per equation, L equations per variable



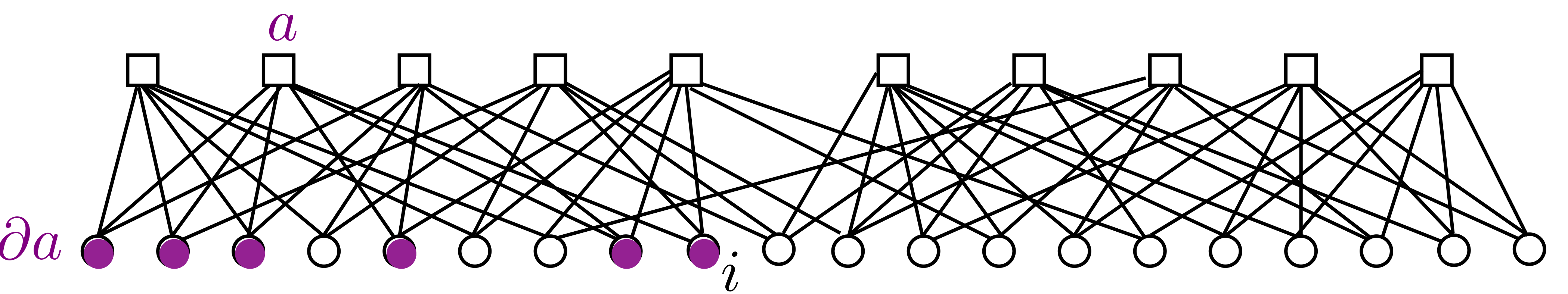
Pb: how to decode?
Inference problem

Error decoding: inference problem



$$P(x_1,\ldots,x_N|y_1,\ldots,y_N) = \frac{1}{Z}\prod_i \psi_i(x_i|y_i)\ \prod_a \mathbb{I}\left(\sum_{i\in\partial a} x_i = 0\ (mod\ 2)\right)$$

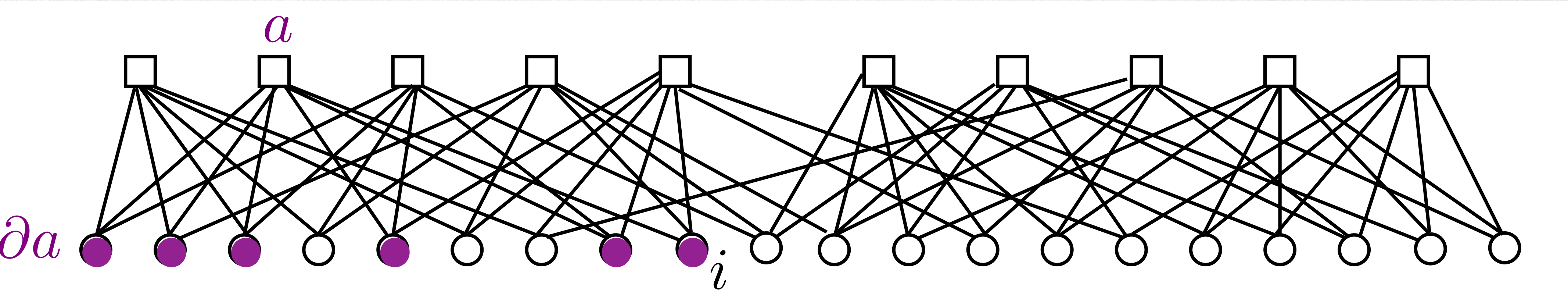
Error decoding: inference problem



$$P(x_1,\ldots,x_N|\boxed{y_1,\ldots,y_N}) = \frac{1}{Z}\prod_i \psi_i(x_i|y_i) \prod_a \mathbb{I}\left(\sum_{i\in\partial a} x_i = 0 \ (mod\ 2)\right)$$

received

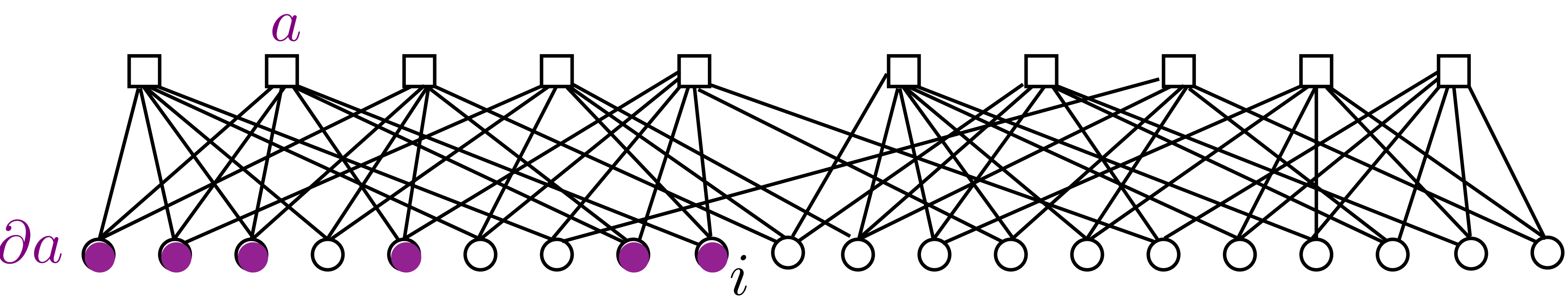
Error decoding: inference problem



$$P(x_1,\ldots,x_N|\underbrace{\boxed{y_1,\ldots,y_N}}_{\text{received}}) = \frac{1}{Z}\boxed{\prod_i \psi_i(x_i|y_i)}\ \prod_a \mathbb{I}\left(\sum_{i\in\partial a} x_i = 0\ (mod\ 2)\right)$$

A priori knowledge
of the channel

Error decoding: inference problem



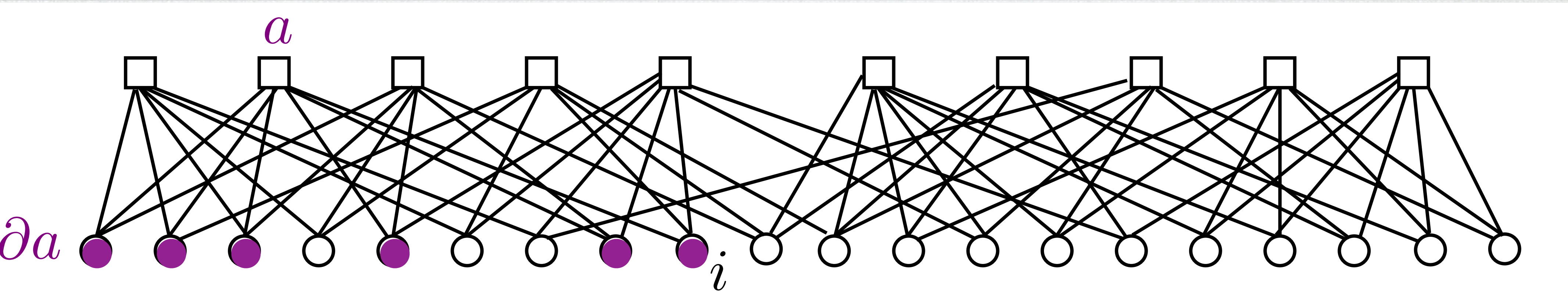
$$P(x_1,\ldots,x_N|\boxed{y_1,\ldots,y_N}) = \frac{1}{Z}\boxed{\prod_i \psi_i(x_i|y_i)}\ \boxed{\prod_a \mathbb{I}\left(\sum_{i\in\partial a} x_i = 0\ (mod\ 2)\right)}$$

received

A priori knowledge of the channel

Parity check constraints

Error decoding: inference problem

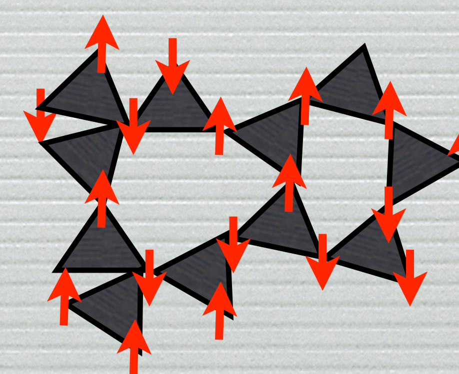


$$P(x_1,\dots,x_N|\underset{\text{received}}{\boxed{y_1,\dots,y_N}}) = \frac{1}{Z}\boxed{\prod_i \psi_i(x_i|y_i)}\ \boxed{\prod_a \mathbb{I}\left(\sum_{i\in\partial a} x_i = 0\ (mod\ 2)\right)}$$

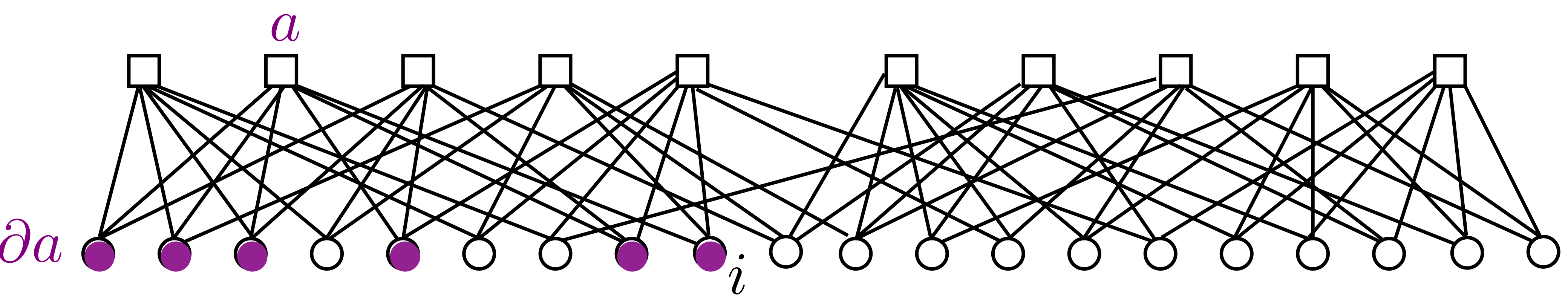
A priori knowledge of the channel

Parity check constraints

Spin glass problem with multispin interactions, discontinuous glass transition (1 step RSB)



Error decoding: inference problem



$$P(x_1, \dots, x_N | y_1, \dots, y_N) = \frac{1}{Z} \prod_i \psi_i(x_i|y_i) \; \prod_a \mathbb{I}\left(\sum_{i \in \partial a} x_i = 0 \; (mod \; 2) \right)$$

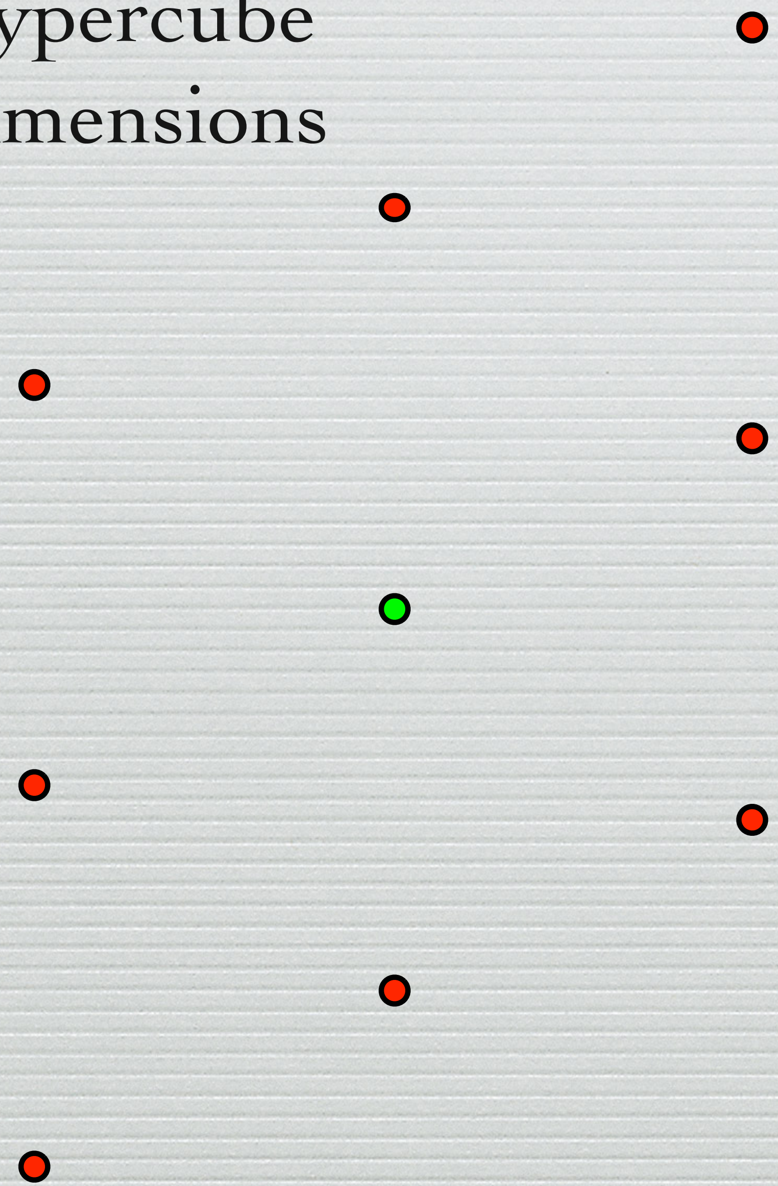
Spin glass problem with multispin interactions

Decoding algo. = iterate mean-field «cavity» equations.
« Message passing » algorithm,

« Belief propagation ». See below

Error correction: decoding

Unit hypercube
in N dimensions

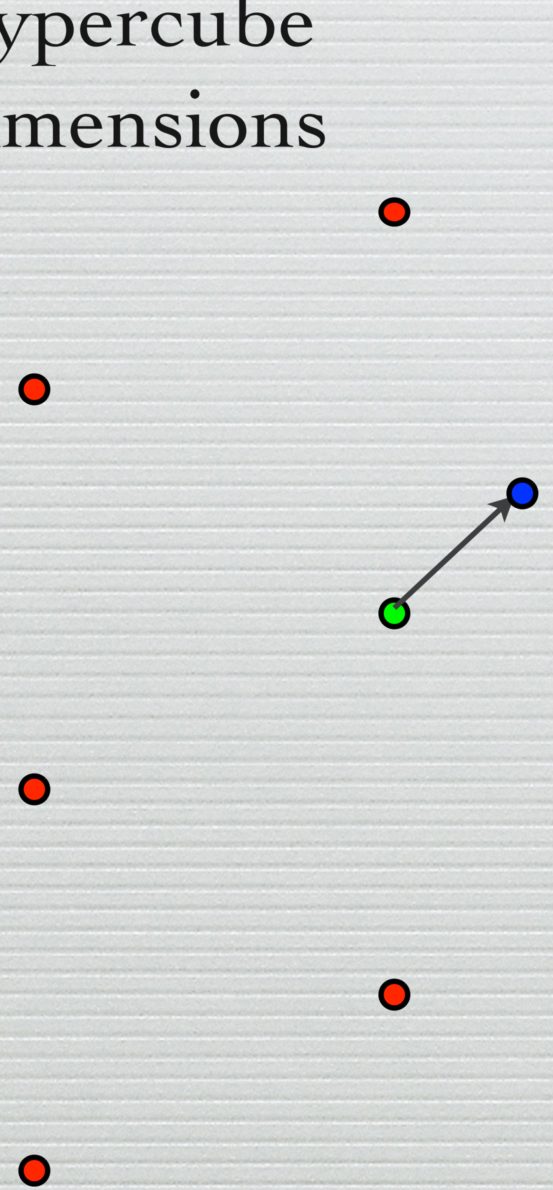


codewords

sent codeword

Error correction: decoding

Unit hypercube
in N dimensions



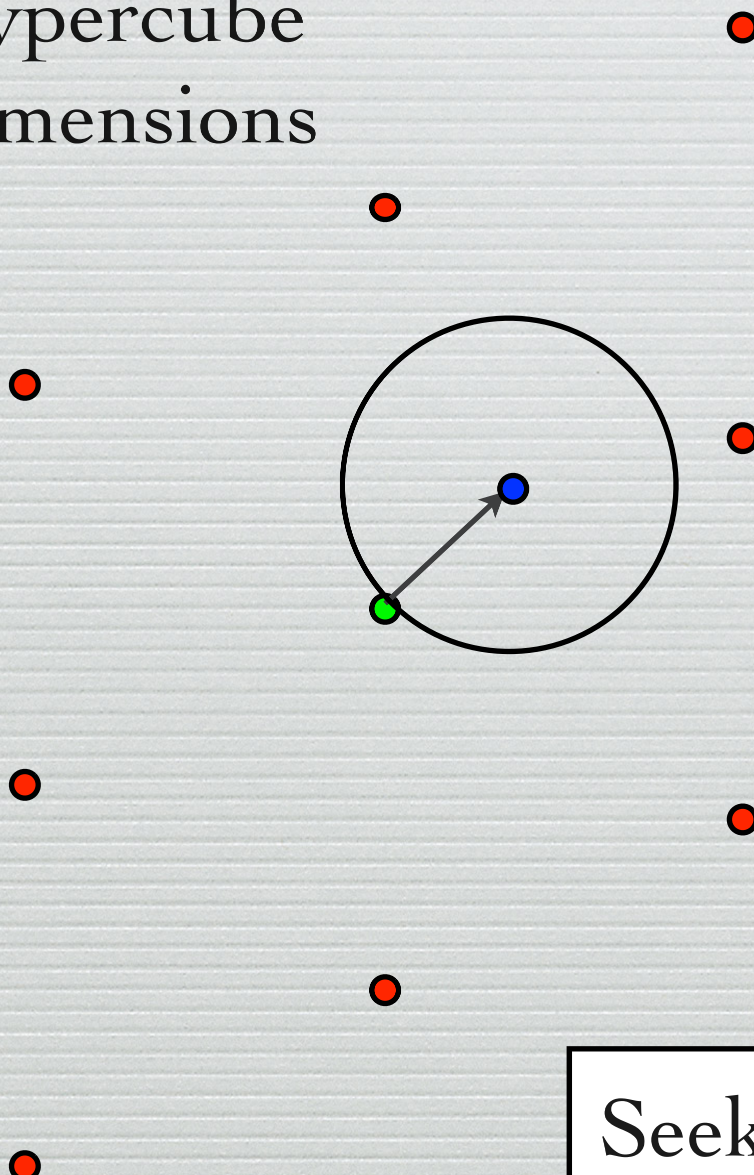
codewords

sent codeword

received word

Error correction: decoding

Unit hypercube
in N dimensions



codewords

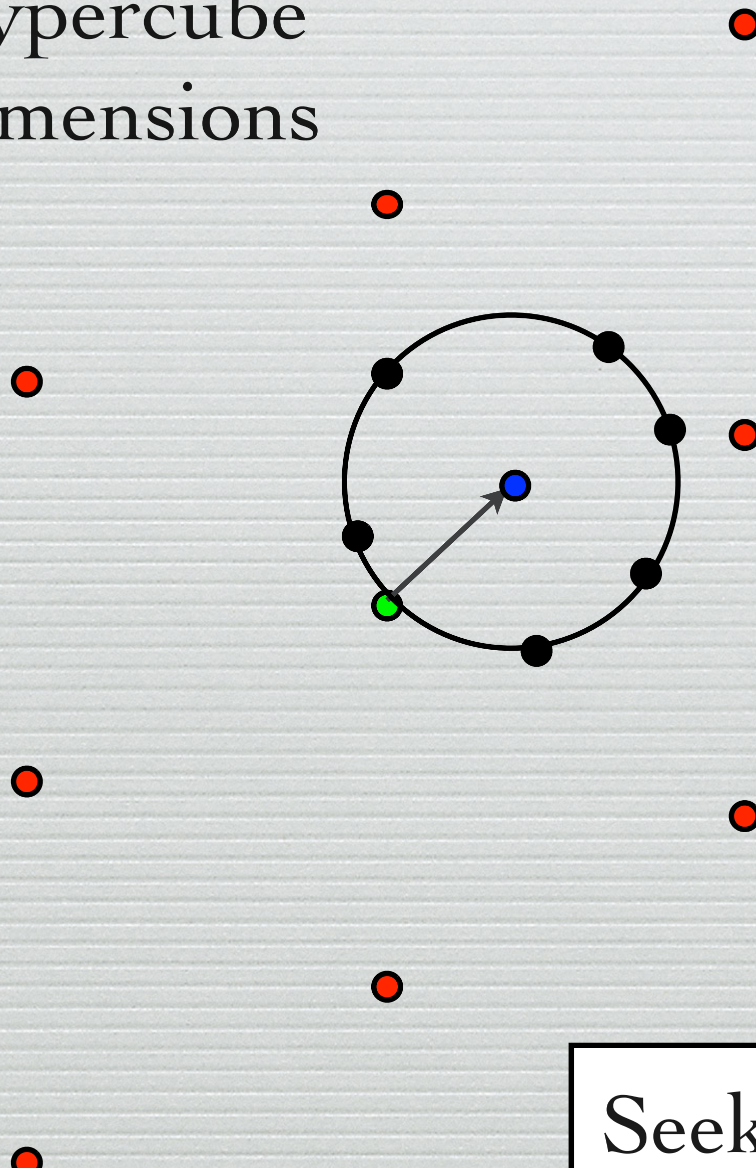
sent codeword

received word

Seek a codeword at distance pN
from received word

Error correction: decoding

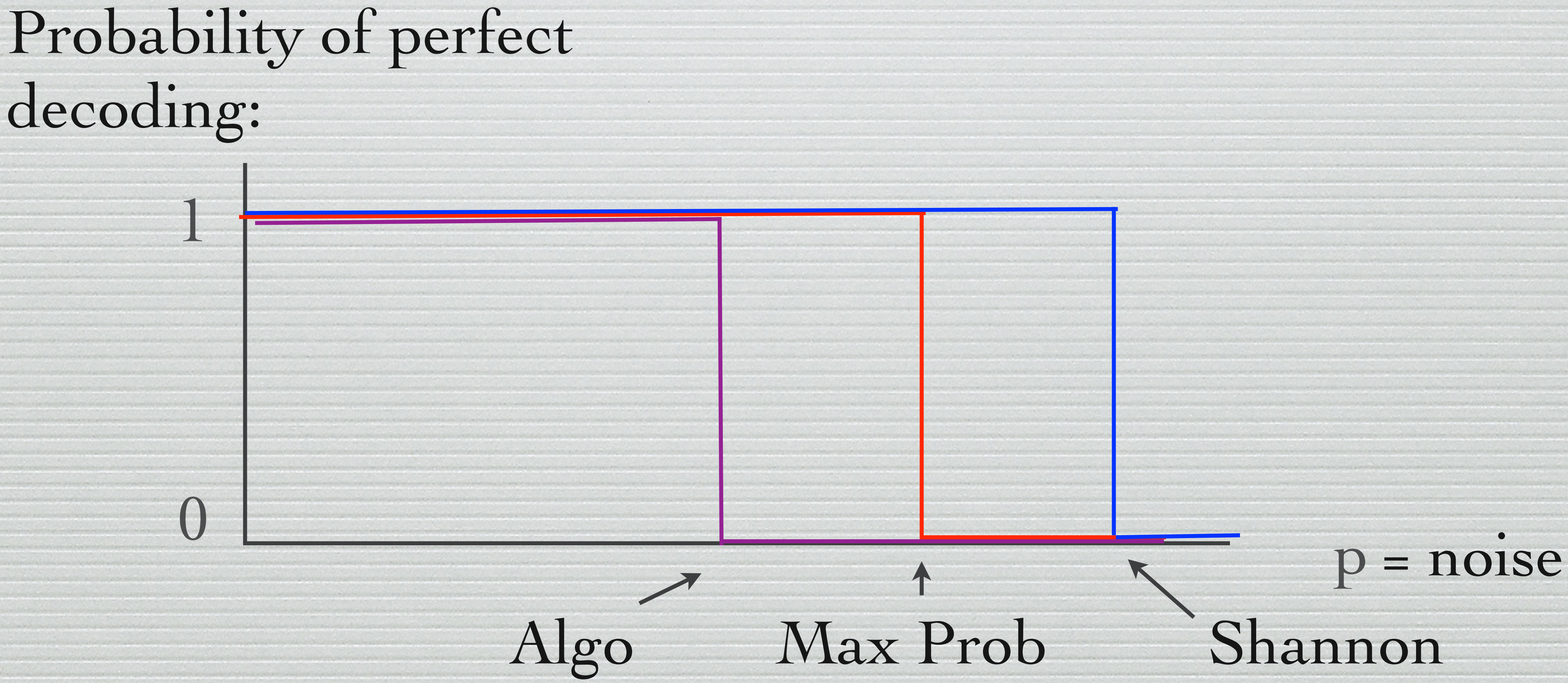
Unit hypercube
in N dimensions



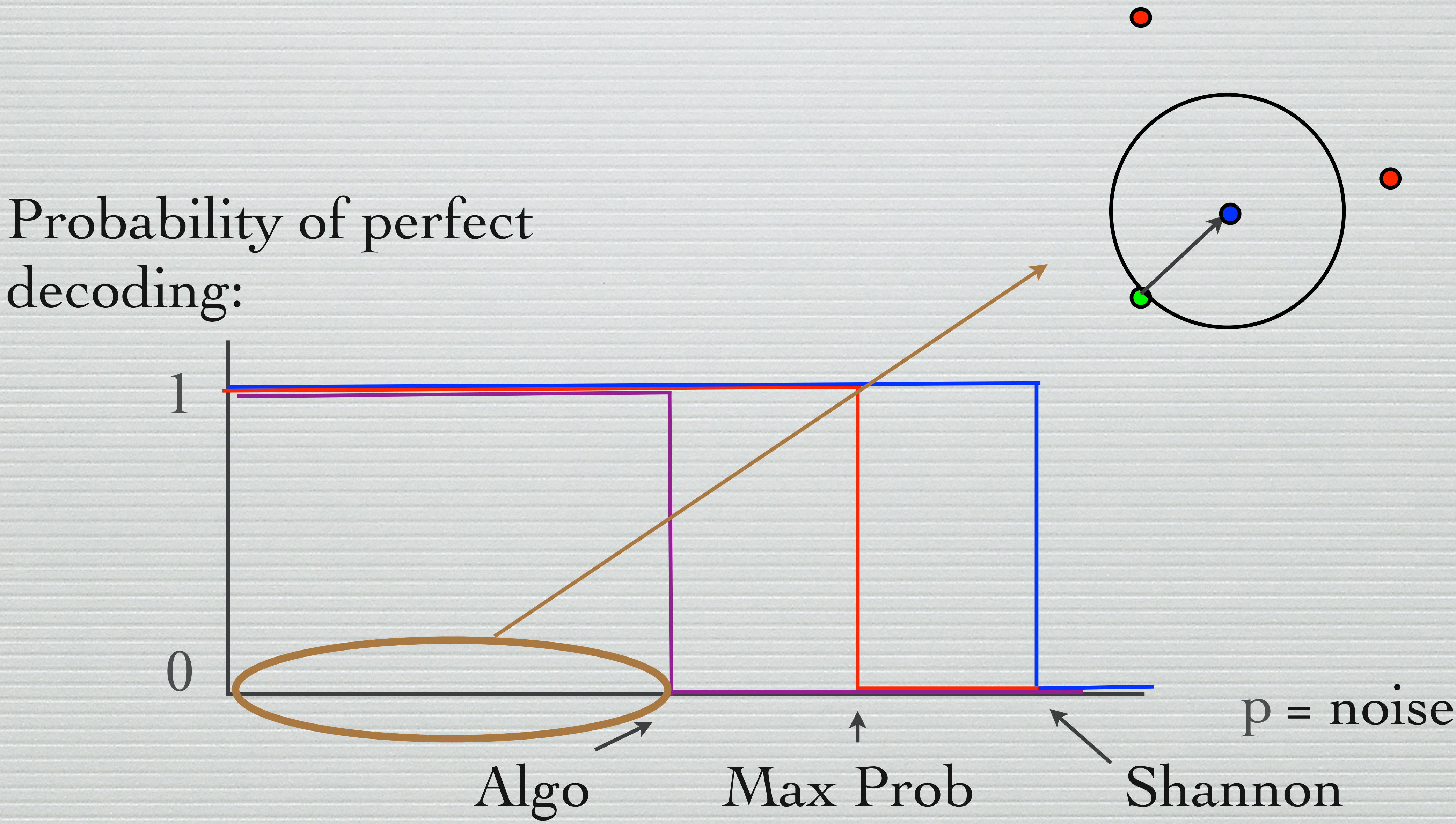
- codewords
- sent codeword
- received word
- metastable states

Seek a codeword at distance pN from received word

Phase transitions in decoding

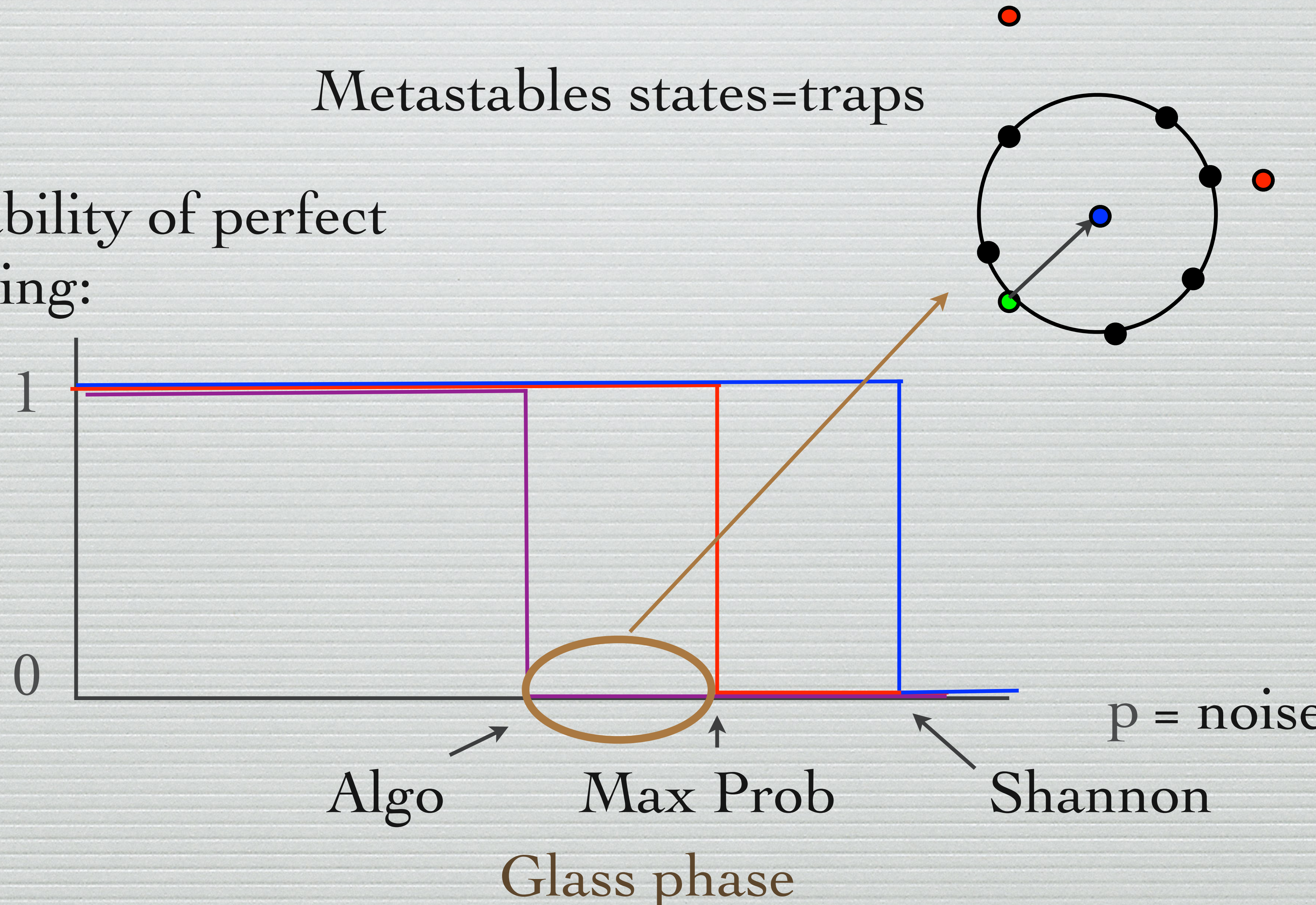


Phase transitions in decoding

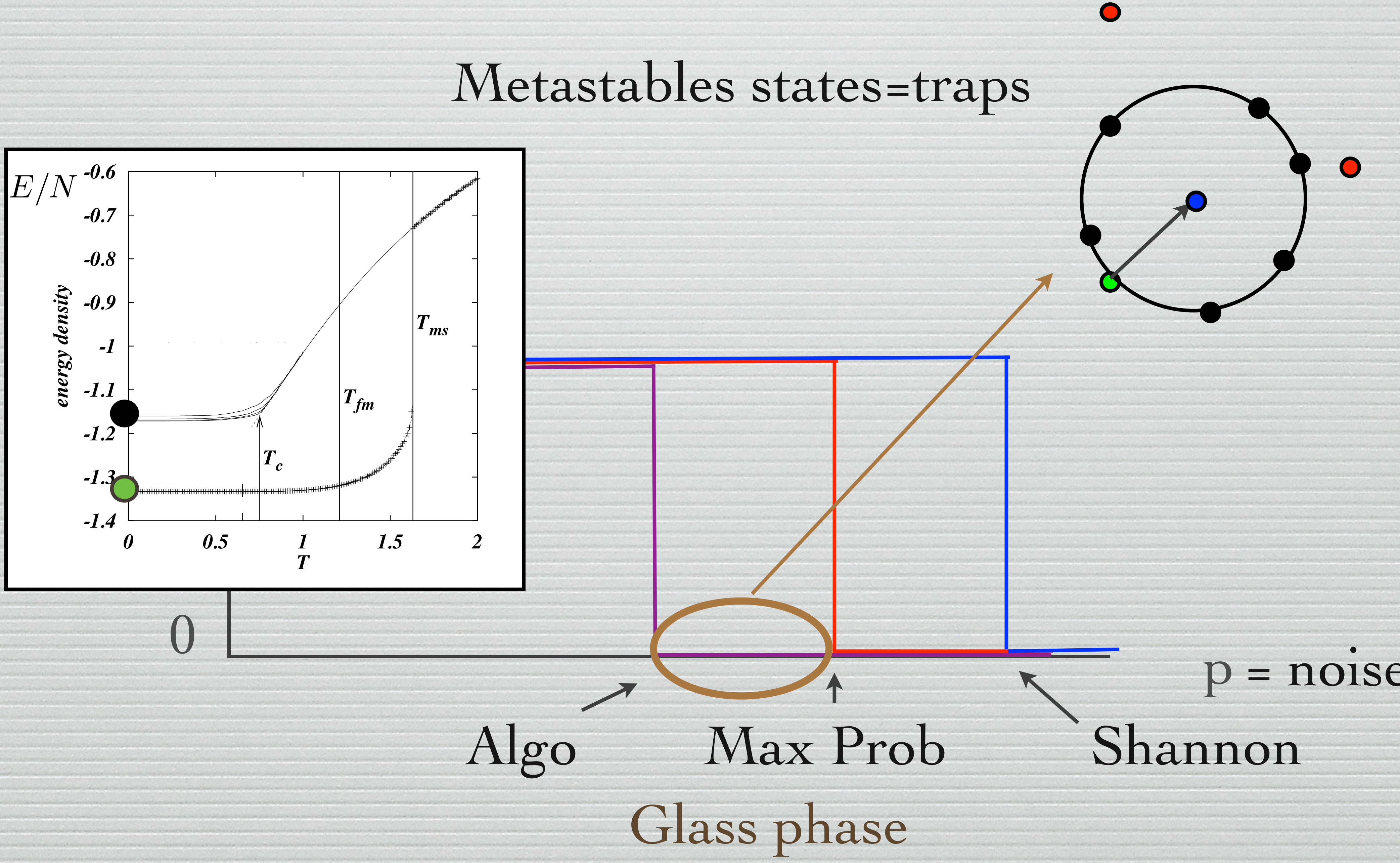


Phase transitions in decoding

Probability of perfect decoding:



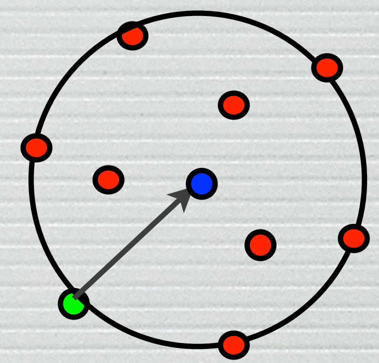
Phase transitions in decoding



Phase transitions in decoding

Phase transitions in decoding

Probability of perfect decoding:

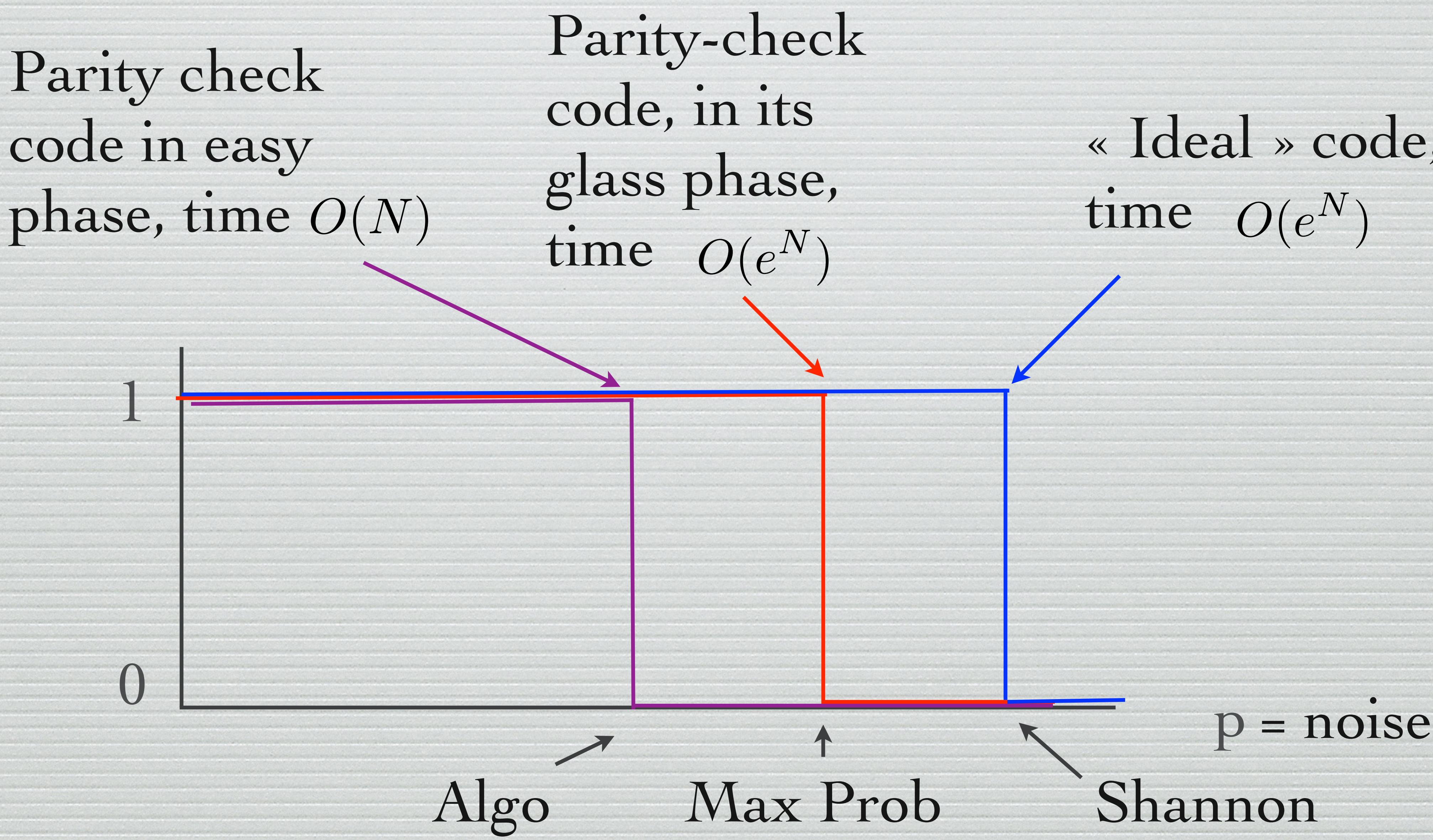


Algo — Max Prob — Shannon

Ex: 3-6 code on BSC

$p_d = .084$ — $p_c = .101$ — $p_S = .110$

Phase transitions in decoding



An example from computer science: Random Satisfiability

N Binary variables $\quad x_i \in \{0,1\}$

M Constraints = clauses, e.g.: $\qquad x_1 \vee \overline{x}_2 \vee x_3$

Is there a configuration of the $\{x_i\}$ which satisfies all the constraints?

The grandfather of NP-complete problems. CNF

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The grandfather of NP-complete problems. CNF

k-SAT (clauses of length $k \geq 3$) is also NP-complete

$\boxed{\text{Typically hard instances: random k-SAT:}}$ Generate each clause with three randomly chosen variables in $\{x_i, \overline{x}_i\}$

Ensemble

Phase transition in the random k-SAT ensemble

Random k-SAT: N variables, M clauses. k variables in each clause, randomly chosen, randomly negated:

Large N limit: $\alpha = M/N$

=density of constraints

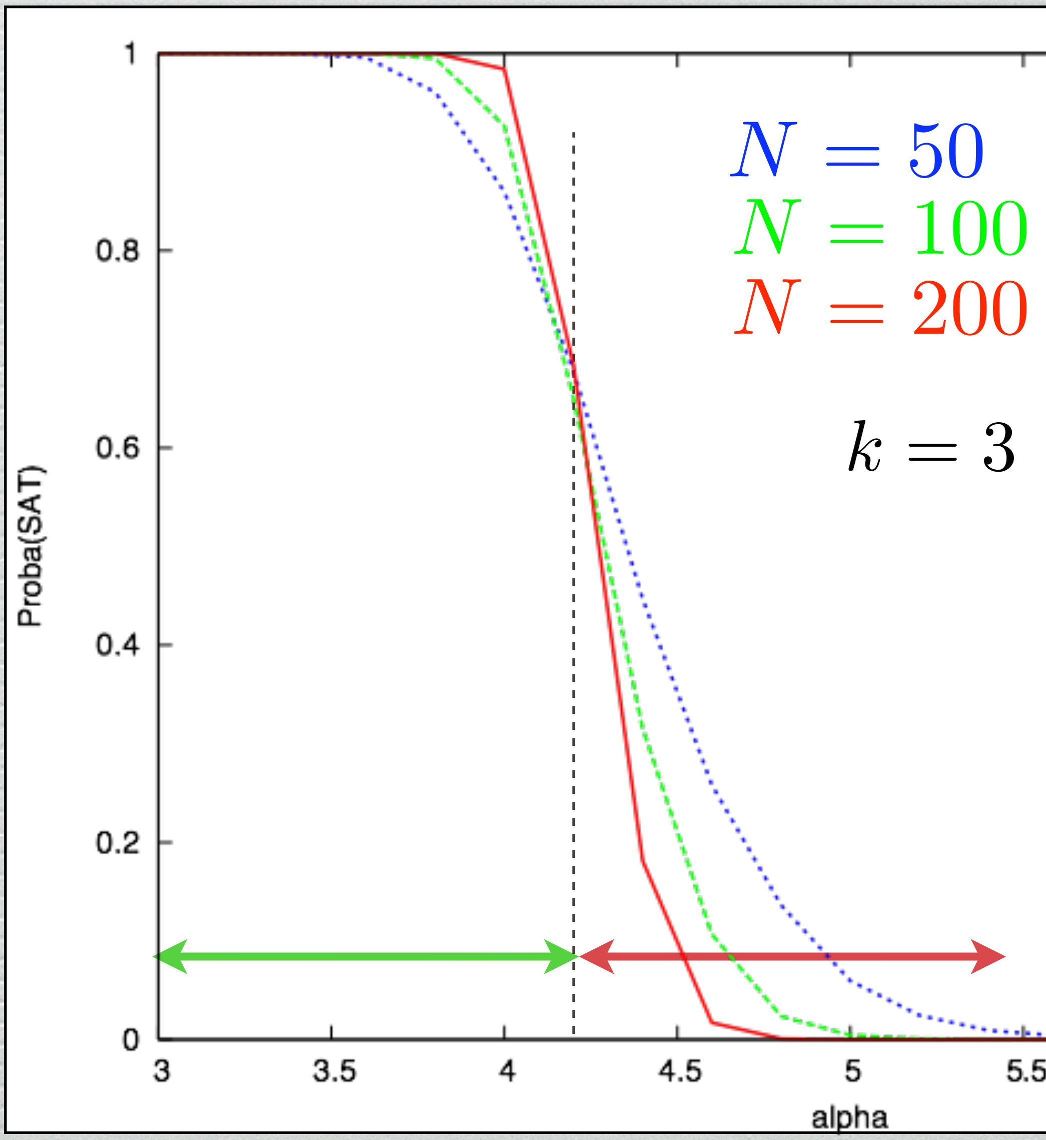
Phase transition

SAT for $\alpha < \alpha_s$

UNSAT for $\alpha > \alpha_s$

Stat-Phys analysis from cavity method -MM Parisi Zecchina (2002): phase transition and algorithm

Proven for k large enough by Ding-Sly-Sun (2015), making rigorous the stat phys approach



Statistical physics of satisfiability

- many binary variables $x = (x_1, \cdots x_N), \; N \gg 1$
- Cost function $E(x)$ = Number of violated constraints = sum of three-body terms
- Find configuration of lowest cost

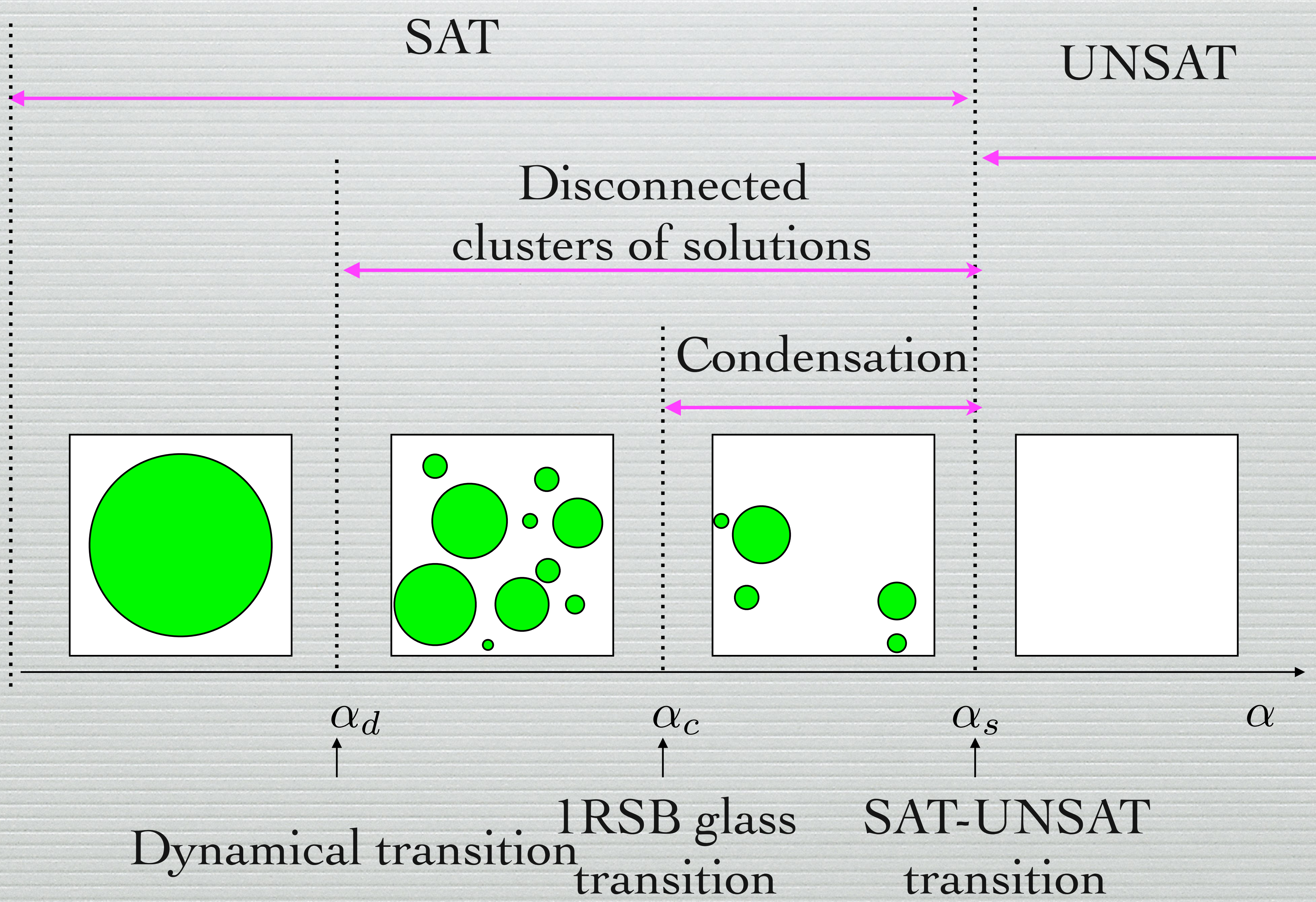
Uniform measure over all SAT assignments

$$P(x) = C\delta_{E(x),0}$$

Kirkpatrick, Selman; Monasson, Zecchina; Biroli, Monasson, Weigt; Mézard, Zecchina; Mézard, Parisi, Zecchina; Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborova; Coja-Oghlan Panagiotou, Ding Sly Sun…

Results

Random k-Satisfiability: clustering



General setup

Phase transitions and glass phases are ubiquitous, in nature and in algorithms trying to solve constraint satisfaction problems with many variables and constraints.

Phase transitions relate to the equilibrium system, reached after infinitely long (i.e. exponential) time

Dynamic glass transitions relate to the practical performance of (polynomial time) algorithms

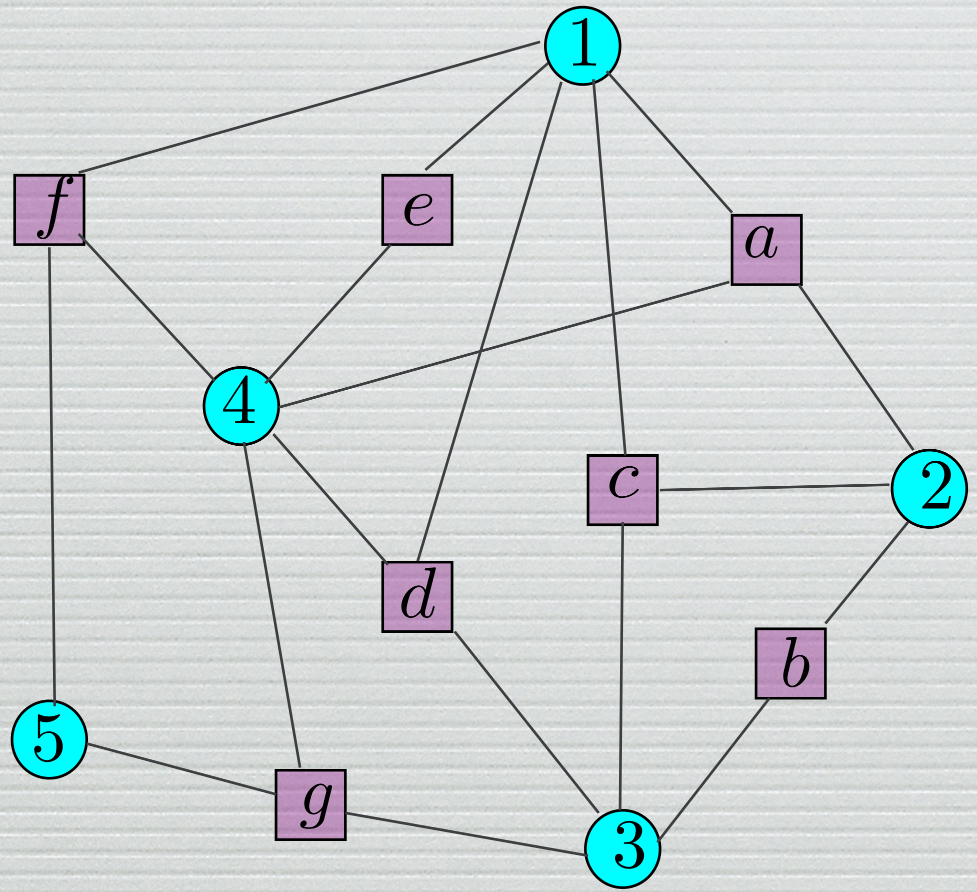
→ Deep links of spin glass theory to information theory and computer science

Cavity method and mean-field based algorithms

Historical development of mean field equations

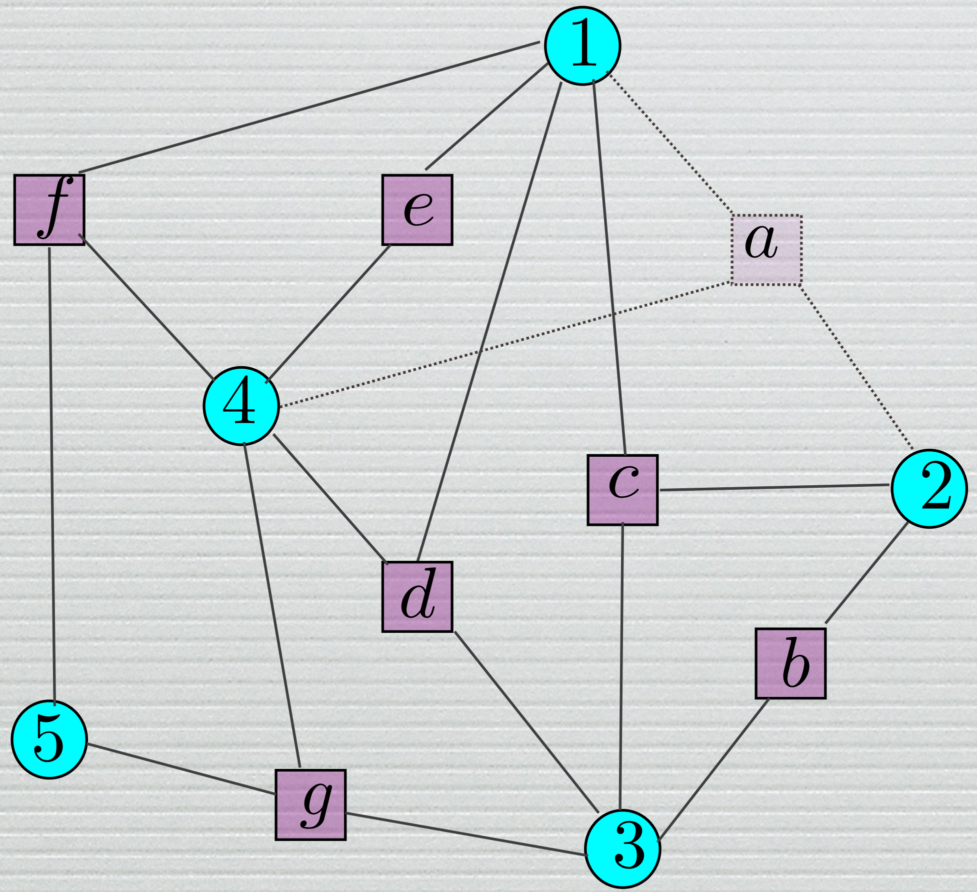
- In homogeneous ferromagnets:
  - Weiss (infinite range, 1907)
  - Bethe Peierls (finite connectivity, 1935)
- In glassy systems:
  - Thouless Anderson Palmer 1977,
  - MM Parisi Virasoro 1986 (infinite range)
  - Kabashima Saad 1998 (finite connectivity)
  - MM Parisi 2001 (finite connectivity)
- As an algorithm:
  - Gallager 1963
  - Pearl 1986
  - MM Parisi Zecchina 2002
  - Kabashima 2003, 2008
  - Donoho Bayati Montanari 2009
  - Rangan 2010
  - Krzakala MM Zdeborova 2012 …

BP = Bethe-Peierls = Belief Propagation



$$P(x_1, \cdots, x_5) = \psi_a(x_1, x_2, x_4)\psi_b(x_2, x_3) \cdots$$

BP equations

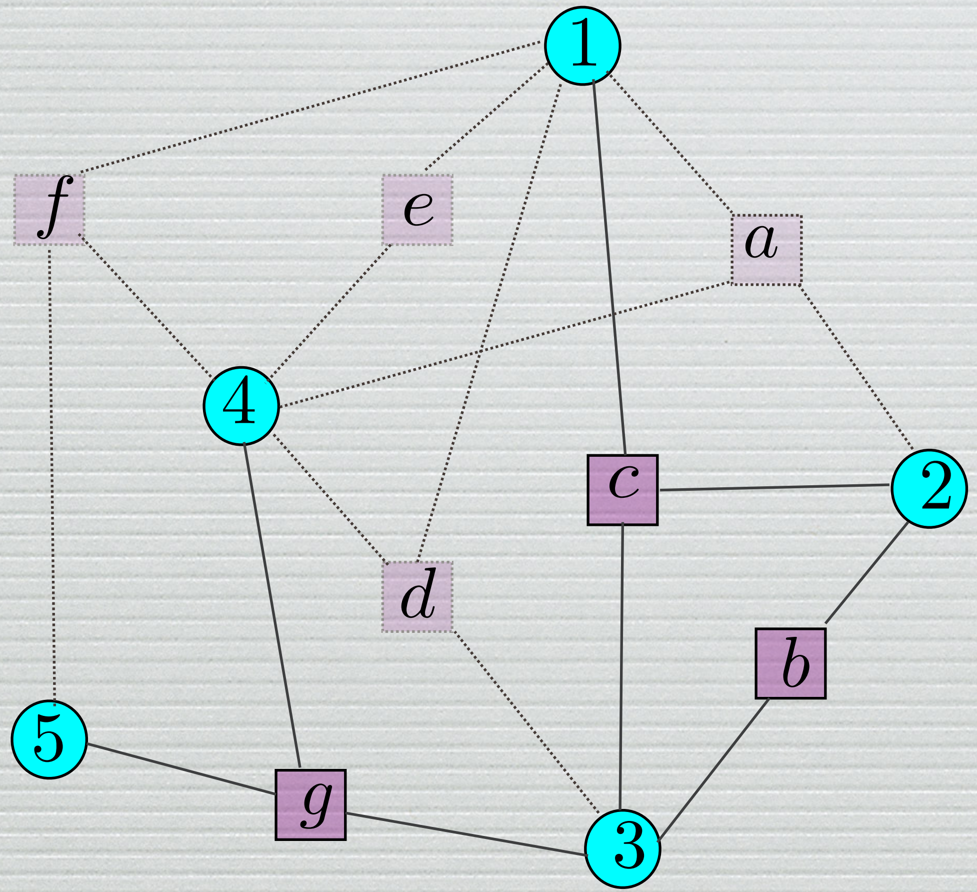


First type of messages:

Probability of x_1 in the absence of a:

$$m_{1\to a}(x_1)$$

BP equations

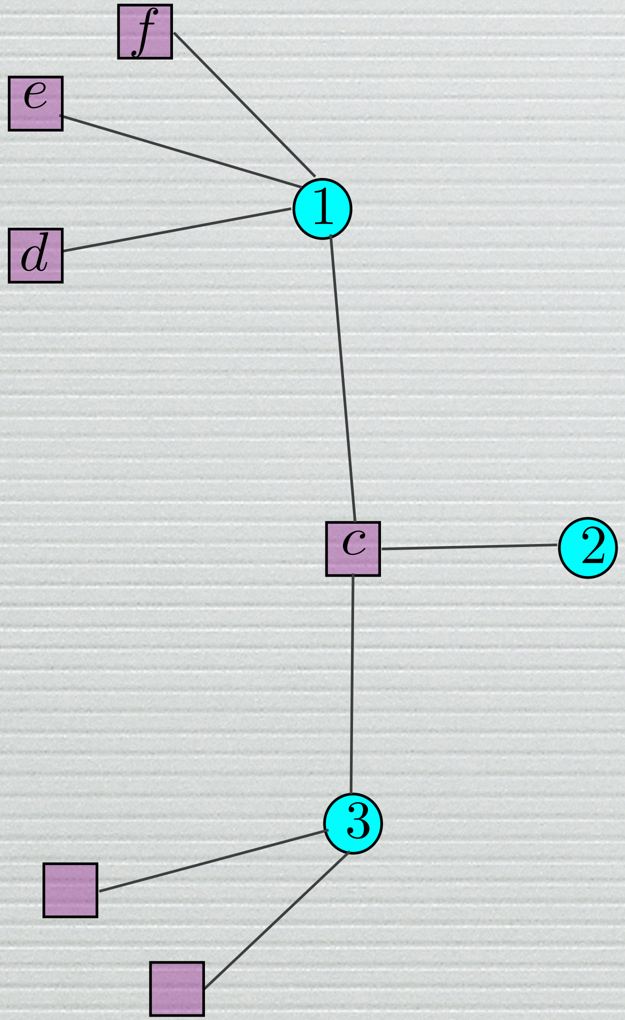


Second type of messages:

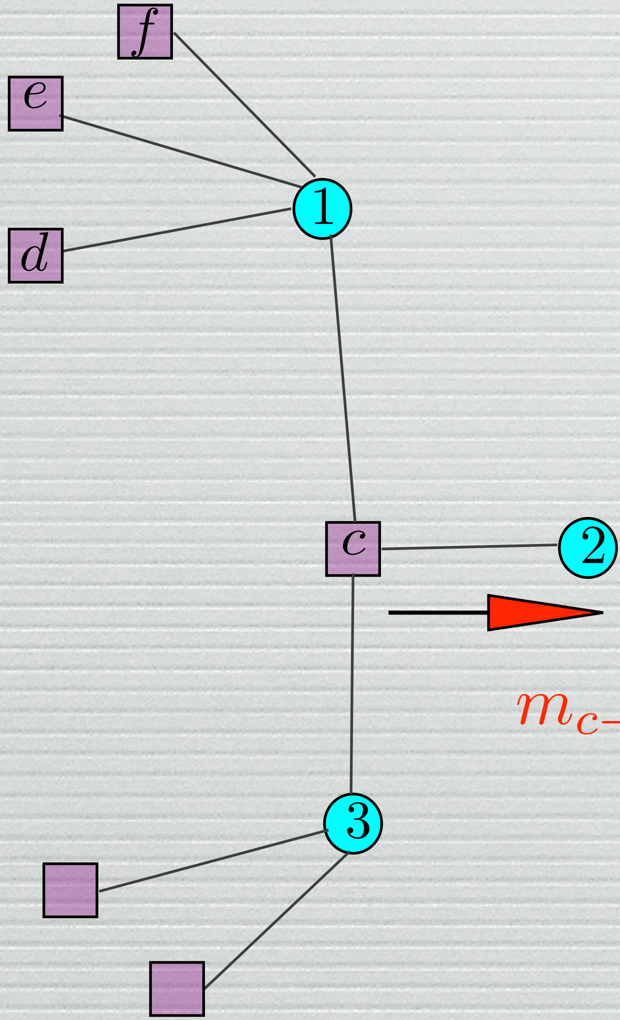
Probability of x_1 when it is connected only to c:

$$m_{c \to 1}(x_1)$$

BP equations

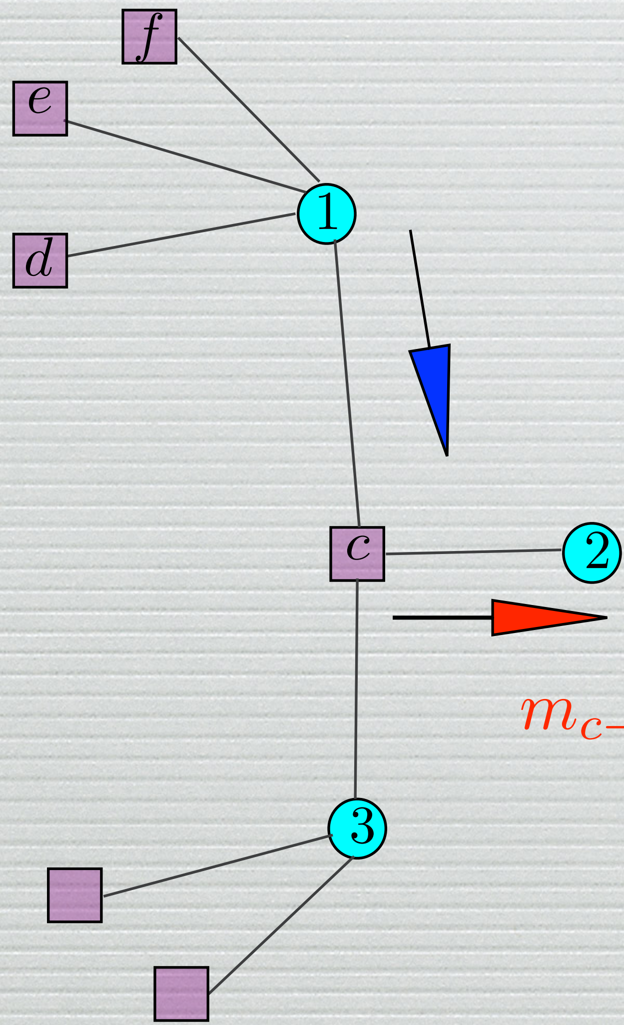


BP equations



$$m_{c\to 2}(x_2) = \sum_{x_1, x_3} \psi_c(x_1, x_2, x_3) m_{1\to c}(x_1) m_{3\to c}(x_3)$$

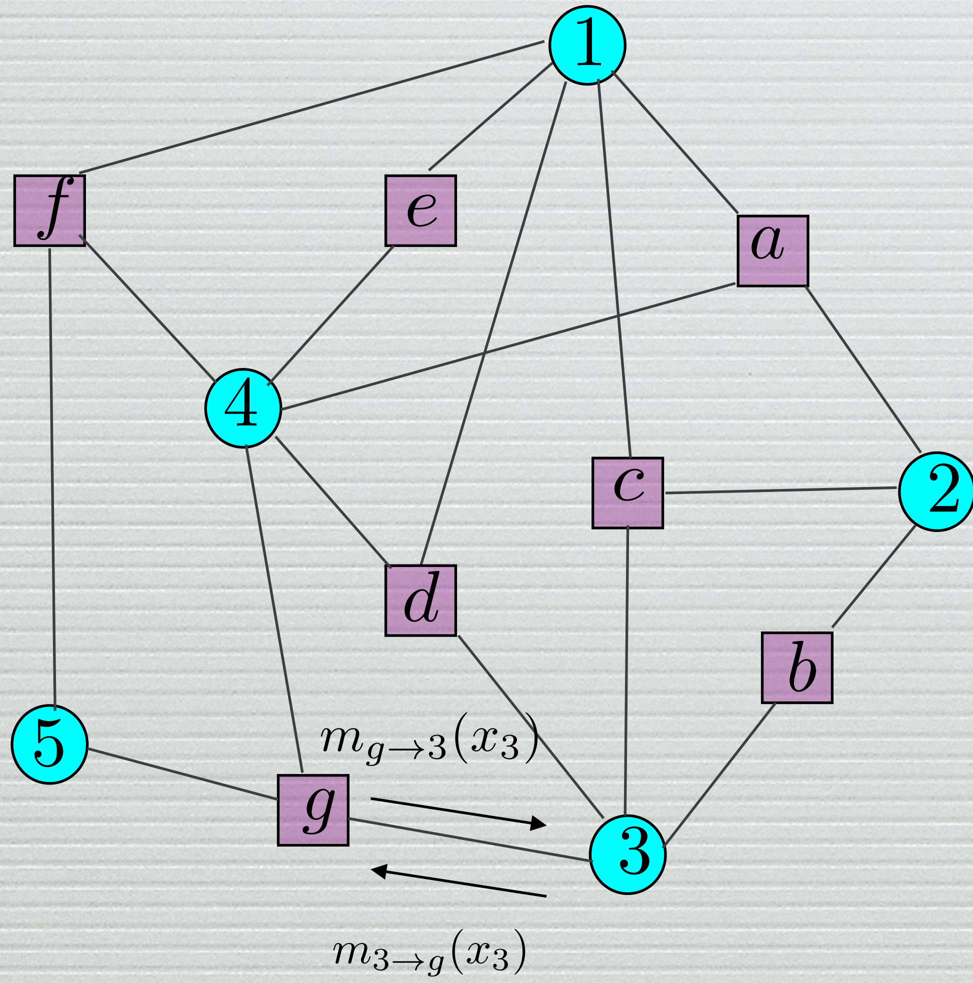
BP equations



$$m_{1\to c}(x_1) = C m_{d\to 1}(x_1) m_{e\to 1}(x_1) m_{f\to 1}(x_1)$$

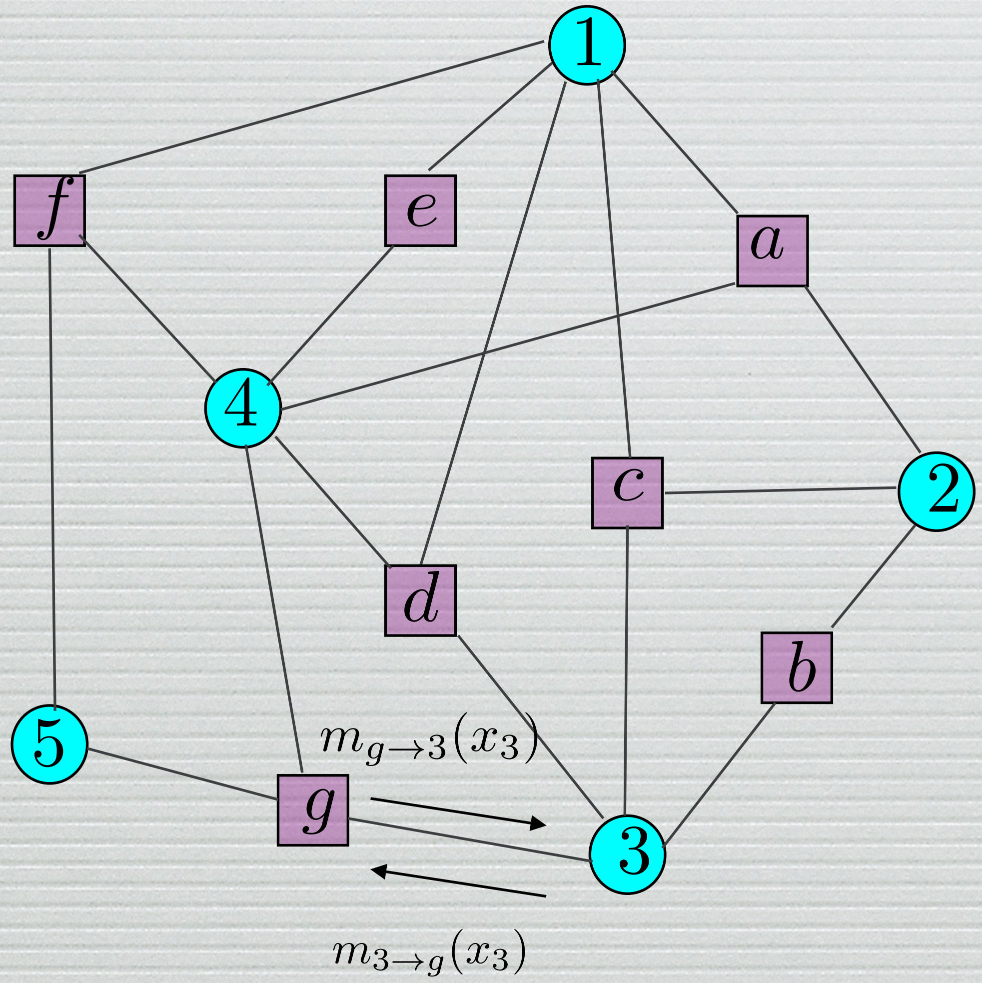
$$m_{c\to 2}(x_2) = \sum_{x_1, x_3} \psi_c(x_1, x_2, x_3) m_{1\to c}(x_1) m_{3\to c}(x_3)$$

BP equations



Propagate messages along the edges, update messages at vertices, using elementary local probabilistic rules

BP equations



Propagate messages along the edges, update messages at vertices, using elementary local probabilistic rules

Closed set of equations: two messages "propagate" on each edge of the factor graph.

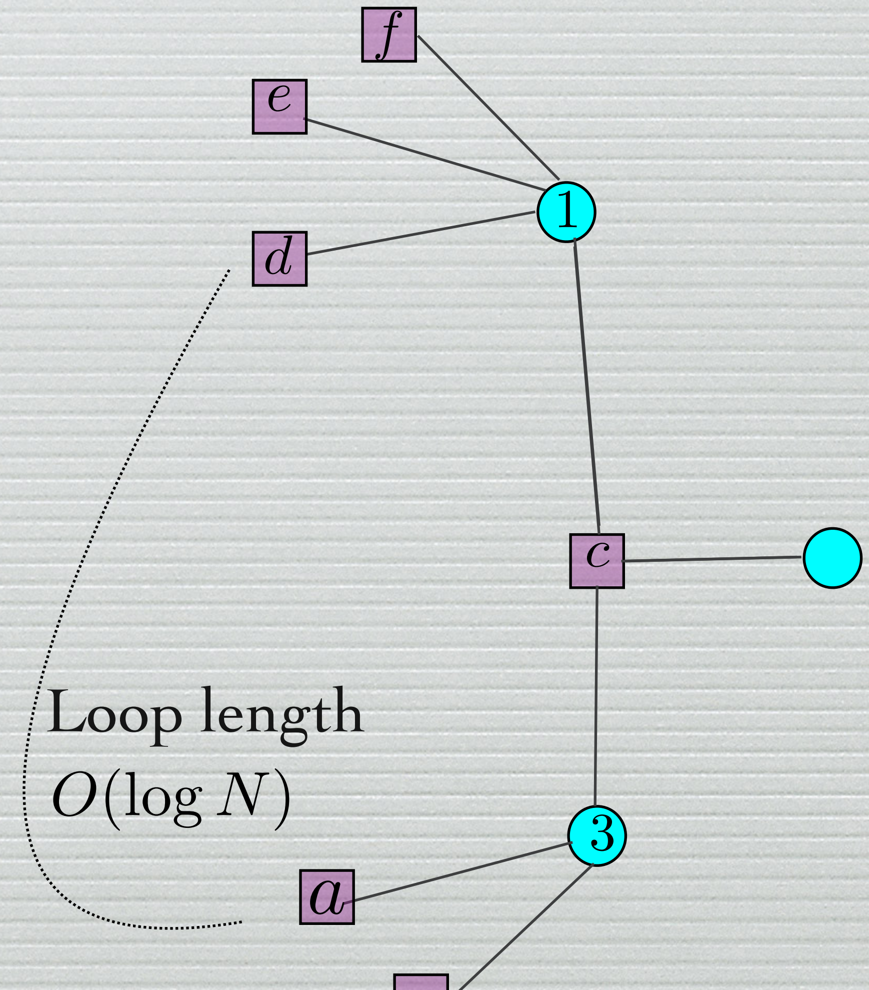
When is BP exact?

$$m_{1\to c}(x_1) = C m_{d\to 1}(x_1) m_{e\to 1}(x_1) m_{f\to 1}(x_1)$$

$$m_{c\to 2}(x_2) = \sum_{x_1,x_3} \psi_c(x_1,x_2,x_3) m_{1\to c}(x_1) m_{3\to c}(x_3)$$

Fluctuations are handled correctly, but beware of correlations

- Exact in one dimension (transfer matrix = dynamic programming)
- Exact on a tree (uncorrelated b.c)
- Exact on locally tree-like graphs (Erdös Renyi etc.) if correlations decay fast enough (single pure state) and uncorrelated disorder
- Exact in infinite range problems if correlations decay fast enough (single pure state) and uncorrelated disorder

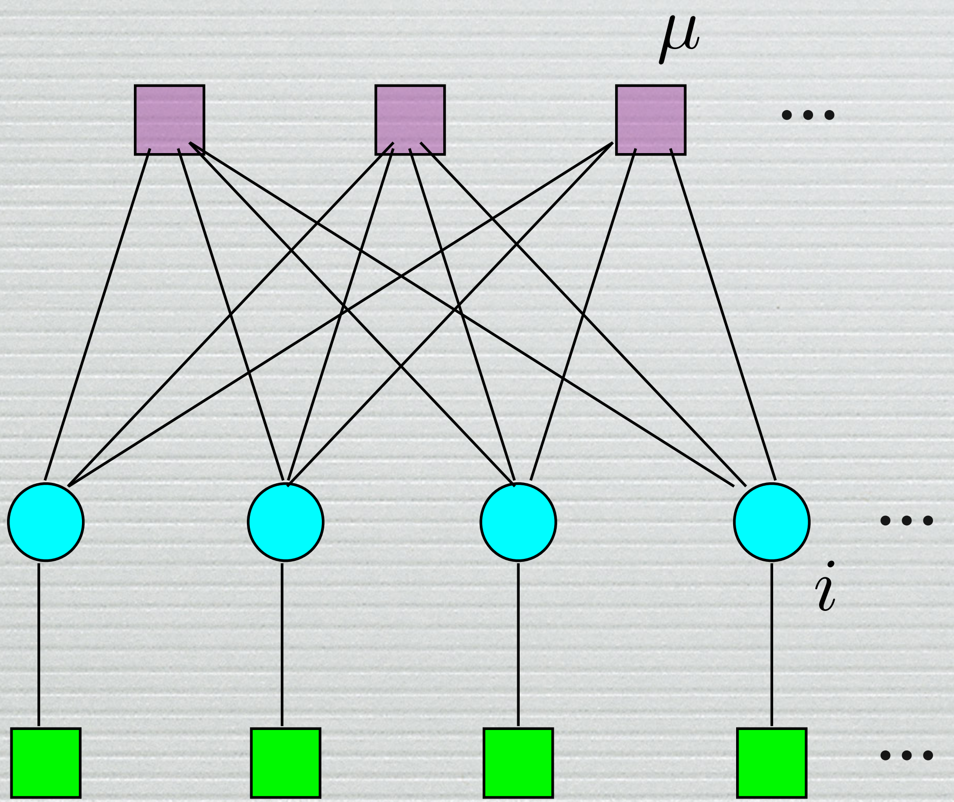


Two important developments

1) The special case of infinite-range models

2) What happens in a glass phase, when there are many pure states, and therefore many solutions ?

Infinite range models : from N^2 messages on the edges to N distributions on the vertices



$$m_{i\to\mu}(x_i) = \prod_{\nu(\neq\mu)} m_{\nu\to i}(x_i)$$

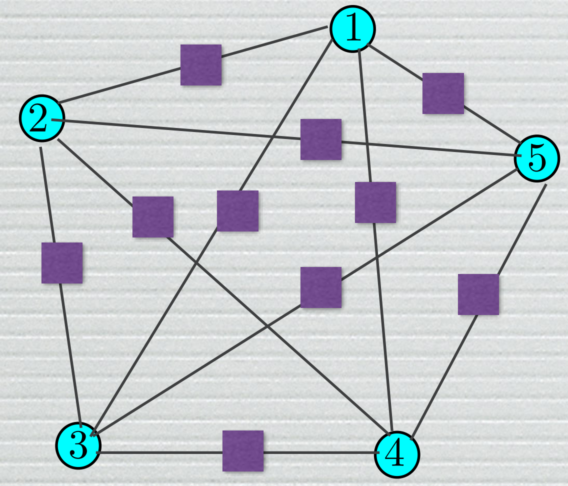
$$M_i(x_i) = \prod_{\nu} m_{\nu\to i}(x_i)$$

Small difference, treated perturbatively

Mean-field equations can be written only in terms of site pdfs: $M_i(x_i)$. TAP, AMP...

Example: SK model

SK model Pairwise interactions $J_{ij} = O\left(\frac{1}{\sqrt{N}}\right)$



$$m_{i\to(ij)}(s_i) \propto e^{h_{i\to j} s_i}$$

$h_{i\to j}$: local field on i in absence of j

BP equations: $$h_{i\setminus j} = \frac{1}{\beta}\sum_{k(\neq i)} \text{atanh}[\tanh(\beta J_{ki})\tanh(\beta h_{k\setminus i})]$$
$$\simeq \sum_{k(\neq i)} J_{ki}\tanh(\beta h_{k\setminus i})$$

Full local field: $$H_i = \frac{1}{\beta}\sum_{k} \text{atanh}[\tanh(\beta J_{ki})\tanh(\beta h_{k\setminus i})]$$
$$\simeq \sum_{k} J_{ki}\tanh(\beta h_{k\setminus i})$$

$h_{i\setminus j} \simeq H_i - O\left(\frac{1}{\sqrt{N}}\right)$: expand in the difference

SK model, TAP equations

SK model Pairwise interactions $J_{ij} = O\left(\frac{1}{\sqrt{N}}\right)$

Corrections can be handled to first order in perturbation theory, and all the equations close on the N variables H_i → **TAP equations**

$$H_i = \sum_k J_{ki} \tanh(\beta H_k) - \beta \tanh(\beta H_i) \sum_k J_{ki}^2 [1 - \tanh^2(\beta H_k)]$$

SK model, TAP equations

SK model Pairwise interactions $J_{ij} = O\left(\frac{1}{\sqrt{N}}\right)$

Corrections can be handled to first order in perturbation theory, and all the equations close on the N variables H_i → **TAP equations**

$$\overset{t+1}{H_i} = \sum_k J_{ki} \tanh(\beta \overset{t}{H_k}) - \beta \tanh(\beta \overset{t-1}{H_i}) \sum_k J_{ki}^2 [1 - \tanh^2(\beta \overset{t-1}{H_k})]$$

Time iteration (Bolthausen): AMP algorithm in information theory

Two important developments

1) The special case of infinite-range models

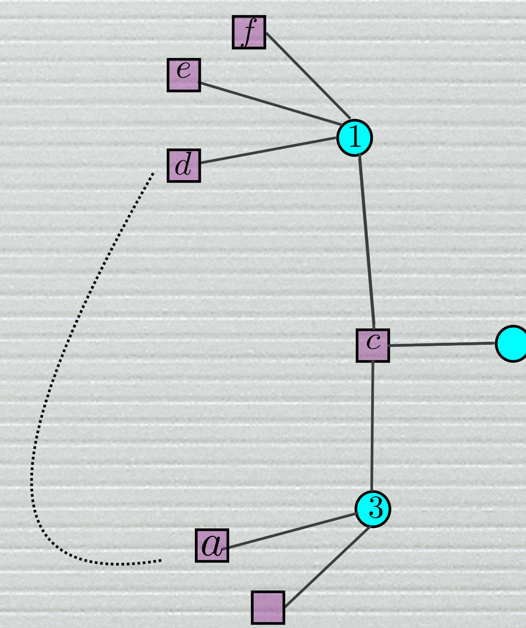
2) What happens in a glass phase, when there are many pure states, and therefore many solutions ?

2) What happens in a glass phase, when there are many pure states, and therefore many solutions ?

BP equations

$$m_{i\to\mu}(x_i) = \prod_{\nu(\neq\mu)} m_{\nu\to i}(x_i)$$

Correct if, in absence of the i-j interaction, the correlations between k and ℓ can be neglected.



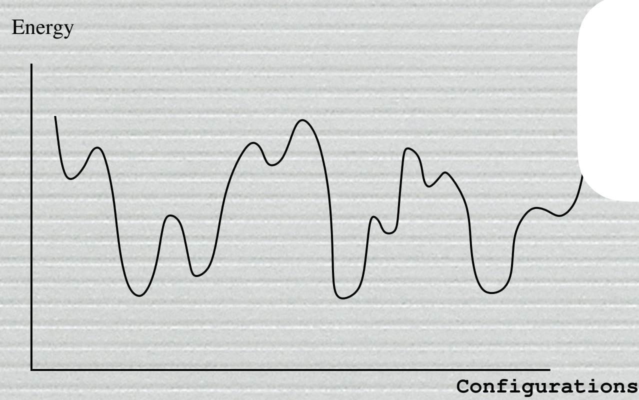
Loop length $O(\log N)$

2) What happens in a glass phase, when there are many pure states, and therefore many solutions ?

BP equations

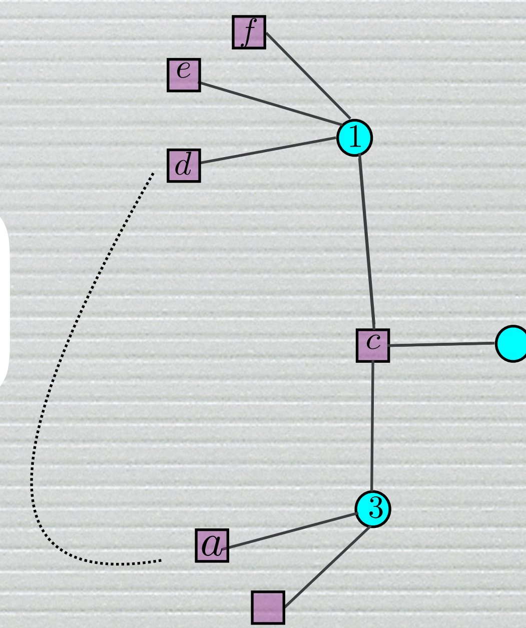
$$m_{i\to\mu}(x_i) = \prod_{\nu(\neq\mu)} m_{\nu\to i}(x_i)$$

Correct if, in absence of the i-j interaction, the correlations between k and ℓ can be neglected.



$$m^{\alpha}_{i\to\mu}(x_i) = \prod_{\nu(\neq\mu)} m^{\alpha}_{\nu\to i}(x_i)$$

Glassy phase: many states, many solutions of BP



Loop length $O(\log N)$

2) What happens in a glass phase, when there are many pure states, and therefore many solutions ?

BP equations

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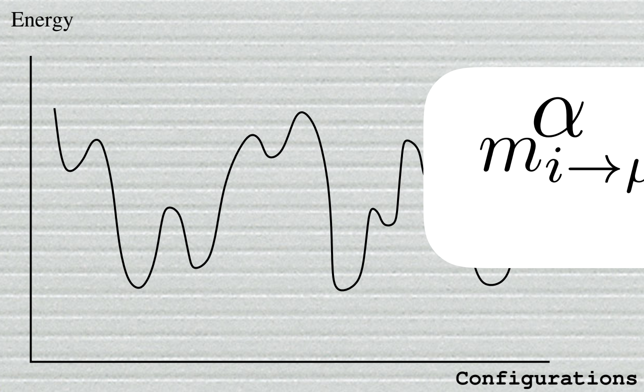


$$m^{\alpha}_{i\to\mu}(x_i) = \prod_{\nu(\neq\mu)} m^{\alpha}_{\nu\to i}(x_i)$$

Glassy phase: many states, many solutions of BP

Statistics of $m^{\alpha}_{i\to\mu}(x_i)$ over the many states α

$$P_{i\to\mu}(m)$$

related to

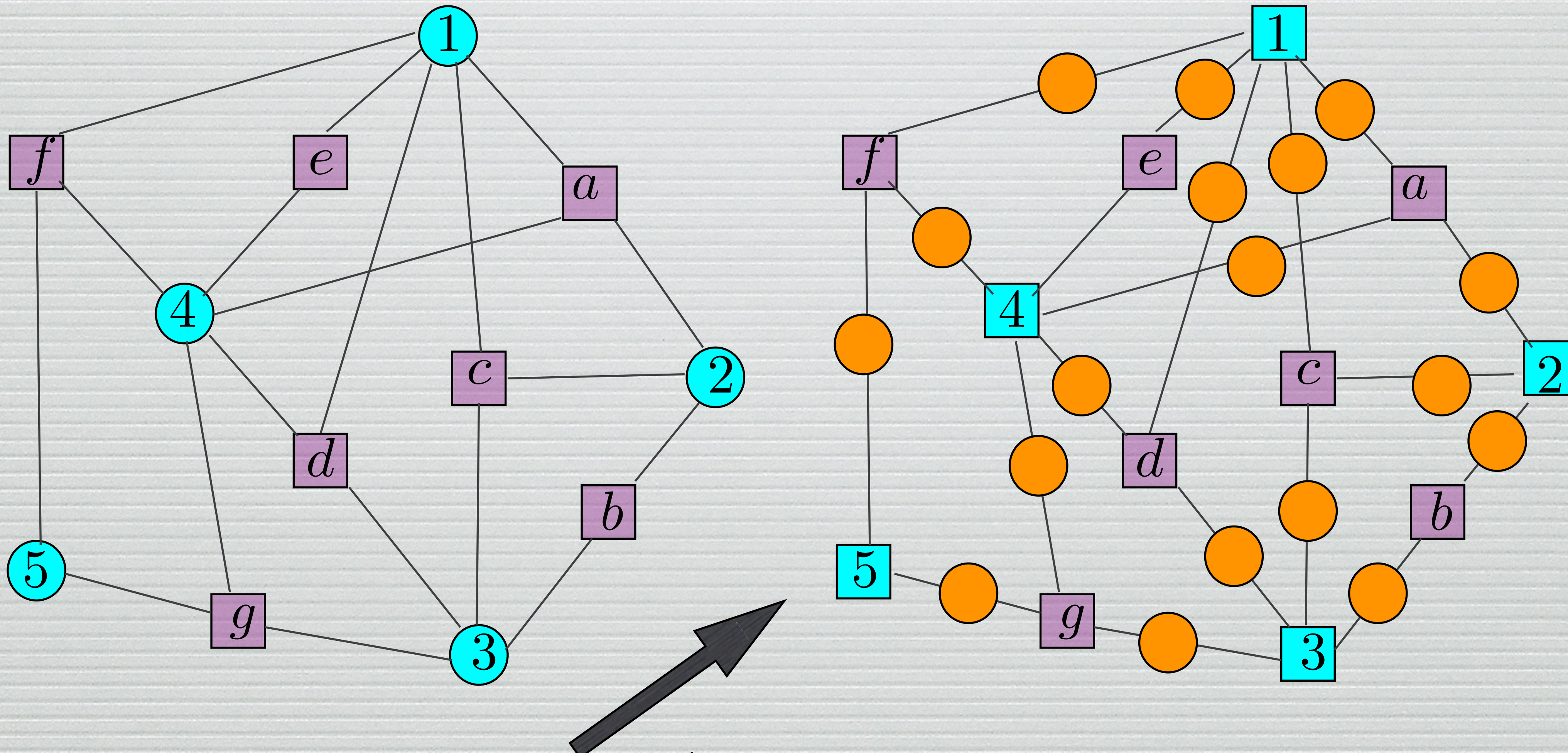
$$P_{\nu\to i}(m)$$

Survey propagation

M Parisi Zecchina 2002

SP=BP ^{2}

Auxiliary problem: statistics over solutions:

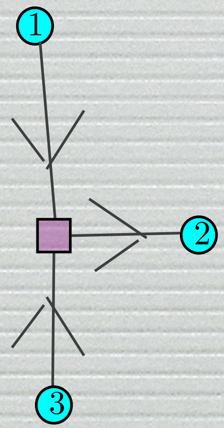


$$P(\{m_{a\to i}\}\{m_{j\to b}\}) = \frac{1}{Z}\prod_{ai}\mathbb{I}(m_{a\to i} = f(\{m_{j\to a}\}) \prod_{jb}\mathbb{I}(m_{j\to b} = f(\{m_{c\to j}\})$$

Power of message passing algorithms

Approximate solution of very hard, and very large constraint satisfaction problems, ...FAST! (typically linear time)

- BP: Best decoders for LDPC error correcting codes
- SP: Best solver of random satisfiability problems
- BP: Best algorithm for learning patterns in neural networks (e.g. binary perceptron)
- Data clustering, graph coloring, Steiner trees, etc…
- Fully connected networks : TAP (=AMP). Compressed sensing, linear estimation, etc.



Local, simple update equations: Each message is updated using information from incoming messages on the same node. Distributed, solves hard global pb

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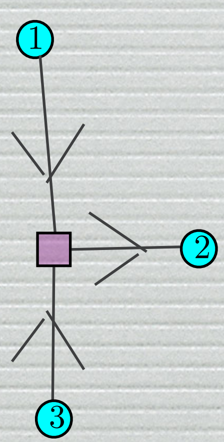

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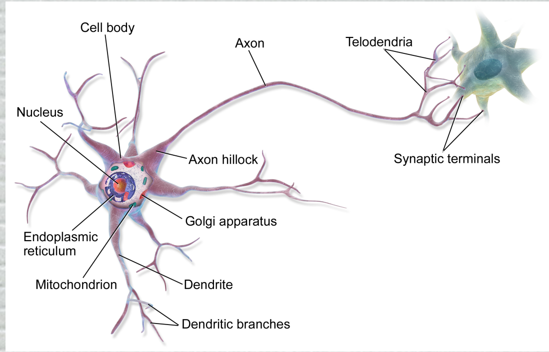
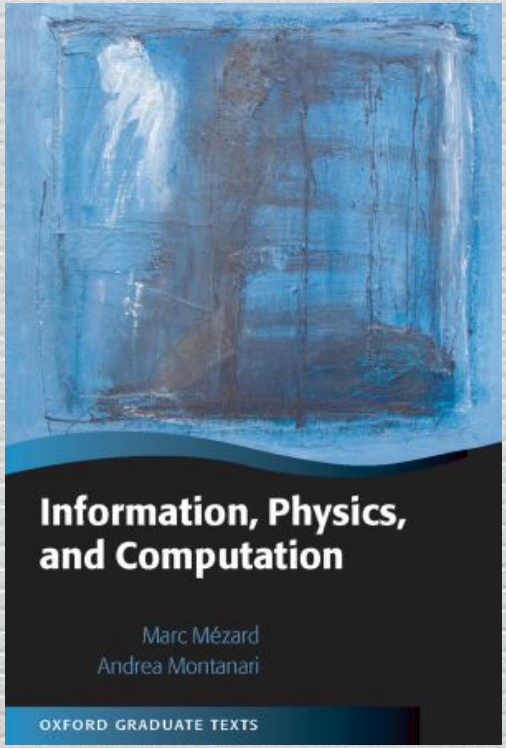


Bird's eye view

« **Representative agent** » mean-field idea is substituted by **statistical approaches to the heterogeneous behaviors of individual agents**

Controlled algorithms, for well chosen ensembles of random problems
Analytic predictions, phase transitions

A very active new field of research, at the interface of statistical physics, information theory and computer science



Physics of glasses : spin, structural, quantum, interfaces, polymers…

Neural networks: brain, capacity, learning…

Mathematics of glasses and constraint satisfaction problems

Inference, statistics, machine learning, proteins, gene expression networks

Economy and finance: portfolio, agent-based models, minority games, order books, risk

Spin glass as a cornucopia

Information theory, codes, signal processing, compressed sensing

Evolution : biological, prebiotic, chemical, self-organization

Optimization and computer science : simulated annealing, quantum annealing, assignment, TSP, K-Sat, BP, SP…

Self-organization, evolution

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Self-organization, evolution

« And it's only the beginning »

Spin glass as a cornucopia: techniques

Replicas,
potential

Cavity:
fully connected, diluted

TAP equations

Ensemble: quenched and
annealed variables

Mathematics:
Interpolation methods
Ultrametricity
Cavity

Spin glass as a cornucopia: concepts

Metastable states

Pure states,
ultrametricity,

Thermodynamic limit
Self-averageness

Condensation of measure,
Poisson-Dirichlet

Glassy dynamics :
landscapes, aging

Link to information
theory: conditional
entropy, mutual
information

Neural networks: brain, capacity, learning…

Inference, statistics, machine learning, proteins, gene expression networks

Statistical physics and statistical inference

Information theory, codes, signal processing, compressed sensing

Next talk

Infer a hidden rule, or hidden variables, from data

Challenge = rules with many hidden parameters (eg deep learning)