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Vector breaking of replica symmetry in some low-temperature disordered systems

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Abstract. We present a new method to study disordered systems in the low-temperature limit. The method uses the replicated Hamiltonian. It studies the saddle points of this Hamiltonian and shows how the various saddle-point contributions can be resummed in order to obtain the scaling behaviour at low temperatures. In a large class of strongly disordered systems, it is necessary to include saddle points of the Hamiltonian which break the replica symmetry in a vector sector, as opposed to the usual matrix sector breaking of the spin glass mean-field theory.

1. Introduction

The use of the replica method has turned out to be very efficient in some disordered systems. It allows a detailed characterization of the low-temperature phase at least at the mean-field level. In all the mean-field spin-glass-like problems where one can expect the mean-field theory to be exact, the Parisi scheme of replica symmetry breaking [1] is successful, and at the moment there is no counterexample showing that it does not work. On the other hand, the low-temperature phase of these systems is complicated enough, even at the mean-field level. One might hope that the very low-temperature limit could be easier to analyse, while its physical content should be basically the same. This very low-temperature limit is also an extreme case where one might hope to get a better understanding of the finite-dimensional problem. At first sight the low-temperature limit is indeed simpler since the partition function could be analysed at the level of a saddle-point approximation. However, it is easy to see that generically this limit does not commute with the limit of the number of replicas going to zero. There is a very basic origin to this non-commutation, namely the fact that there still exist, even at zero temperature, sample-to-sample fluctuations. In this paper we try to develop a method of summation over all saddle points in replica space, in order to get the low temperature behaviour of glassy systems. The main aim of this paper is to propose this new method. We have tested it on some elementary problems which can be solved directly. As for its application to more difficult problems, we have also obtained some very good approximations to the zero temperature fluctuations, a particle in a random medium, as well as some interesting scaling relations in the random field Ising model. Section 2 presents the method and illustrates it on a variety of zero-dimensional problems. In section 3 we discuss the case of directed polymers in random media with long-range interactions, where we rederive the scaling exponents using this new method. In

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section 4 we study the D-dimensional random field Ising model. Perspectives are briefly summarized in section 5.

2. Zero-dimensional systems

2.1. The Ising model

To demonstrate in the simplest terms how the proposed procedure works, we consider first some trivial zero-dimensional problems. The simplest example is one Ising spin $\sigma = \pm 1$ in a random field h. The Hamiltonian is:

$$H = \sigma h \tag{2.1}$$

where the distribution for the random field is Gaussian:

$$P(h) = \frac{1}{\sqrt{2\pi h_0^2}} \exp\left(-\frac{h^2}{2h_0^2}\right). \tag{2.2}$$

The free energy is:

$$-\beta F(h_0;\beta) = \overline{\ln\left[\sum_{\sigma=\pm 1} \exp(-\beta\sigma h)\right]} = \int_{-\infty}^{+\infty} Dx \ln[1 + \exp(2\beta h_0 x)] \tag{2.3}$$

where Dx is the centred Gaussian measure of width one: $Dx = \frac{\mathrm{d}x}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$. In particular, in the zero-temperature limit one finds:

$$F(h_0;\beta\to\infty) = \frac{2h_0}{\sqrt{2\pi}}. \tag{2.4}$$

Let us consider now how this 'problem' can be solved in terms of the replica approach:

$$\begin{aligned}
-\beta F(h_0;\beta) &= \lim_{n\to 0} \frac{1}{n}(\overline{Z^n} - 1) = \lim_{n\to 0} \frac{1}{n}\left[\sum_{\{\sigma_a\}=\pm 1} \exp\left\{\frac{1}{2}\beta^2 h_0^2 \left(\sum_{a=1}^{n} \sigma_a\right)^2\right\} - 1\right] \\
&= \lim_{n\to 0} \frac{1}{n}\left[\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \exp\left\{\frac{1}{2}\beta^2 h_0^2 (2k-n)^2\right\} - 1\right].
\end{aligned} \tag{2.5}$$

In view of the application of the method to more complicated problems we want to compute the behaviour at low temperatures. This cannot be done naively from a saddle-point evaluation of the sum at large β, because of the non-commutativity of the limits $\beta \to \infty$ and $n \to 0$. Instead we proceed as follows. The term $k = 0$, which is the contribution from the 'replica symmetric (RS) configuration' $\sigma_a = +1$, is singled out; its contribution is equal to $1 + \mathrm{O}(n^2)$, which cancels the (-1) in equation (2.5). (Notice that one could also single out the term $k = n/2$, as was done in [2] in another problem; this would lead to the same result.) The contributions of the rest of the terms (which could be interpreted as corresponding to the states with 'replica symmetry breaking' (RSB) in the replica vector $\{\sigma_a\}$) can be represented as follows:

$$F(h_0;\beta) = -\lim_{n\to 0} \frac{1}{\beta n} \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \exp\left\{\frac{1}{2}\beta^2 h_0^2 (2k-n)^2\right\}. \tag{2.6}$$

Here the summation over k can be extended beyond $k = n$ to ∞ since the gamma function is equal to infinity at negative integers.

Now we perform the analytic continuation $n \to 0$, using the relation:

$$\left.\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}\right|_{n\to 0} \simeq n\frac{(-1)^{k-1}}{k}. \tag{2.7}$$

Thus, for the free energy (2.6) one obtains:

$$-\beta F(h_0;\beta) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\exp(2\beta^2 h_0^2 k^2) = \int_{-\infty}^{+\infty} Dx \ln[1+\exp(2\beta h_0 x)]. \tag{2.8}$$

We see that this result coincides with the one (2.3) obtained by the direct calculation. This is of course no surprise since we have just done an exact replica computation. But it exemplifies some of the steps that we shall need below, in particular the proper definition and computation of the divergent series appearing in (2.8) through an integral representation.

2.2. The 'soft' Ising model

Consider now the 'soft' version of the Ising model described by the double-well Hamiltonian:

$$H = -\tfrac{1}{2}\tau\phi^2 + \tfrac{1}{4}\phi^4 - h\phi \tag{2.9}$$

where the random field is described by the Gaussian distribution (2.2). We concentrate again on the zero-temperature limit. Besides, we assume that the typical value of the field h_0 is small ($h_0 \ll \tau^{3/2}$). In this case the field will not destroy the double-well structure of the Hamiltonian (2.9), and (at $T \to 0$) the system must be equivalent to the discrete Ising model considered before. (The 'opposite limit' of the random field Hamiltonian with only one ground state will be considered in section 2.3.)

The direct calculation of the zero-temperature free energy is trivial. For a given value of the field $h \ll \tau^{3/2}$ the ground states of the Hamiltonian (2.9) are: $\phi_1 \simeq +\sqrt{\tau} + h/2\tau$, for $h > 0$; and $\phi_1 \simeq -\sqrt{\tau} + h/2\tau$, for $h < 0$. In both cases the corresponding energy is $E_g(h) \simeq -\frac{1}{4}\tau^2 - |h|\sqrt{\tau}$. Thus, the zero-temperature averaged free energy is:

$$F(h_0) \simeq -\frac{1}{4}\tau^2 - 2\sqrt{\tau}\int_0^{+\infty}\frac{\mathrm{d}h}{\sqrt{2\pi h_0^2}}h\exp\left(-\frac{h^2}{2h_0}\right) = -\frac{1}{4}\tau^2 - \frac{2h_0\sqrt{\tau}}{\sqrt{2\pi}}. \tag{2.10}$$

Consider now how this result can be obtained in terms of replicas. The replica Hamiltonian and the corresponding saddle-point equations are:

$$H_n = -\tfrac{1}{2}\tau\sum_{a=1}^{n}\phi_a^2 + \tfrac{1}{4}\sum_{a=1}^{n}\phi_a^4 - \tfrac{1}{2}\beta h_0^2\left(\sum_{a=1}^{n}\phi_a\right)^2 \tag{2.11}$$

$$-\tau\phi_a + \phi_a^3 = \beta h_0^2\left(\sum_{a=1}^{n}\phi_a\right). \tag{2.12}$$

The RS solution of these equations (in the limit $n \to 0$) is: $\phi_a = \phi_{\mathrm{RS}} = \sqrt{\tau}$. The corresponding energy is $E_{\mathrm{RS}} = -\frac{1}{4}n\tau^2$. This solution (in the limit $n \to 0$) does not involve the contribution from the random field.

Proceeding along the lines of the section 2.1, we have to look also for the solution of equation (2.12) which would involve the RSB in the replica vector $\{\phi_a\}$:

$$\phi_a = \begin{cases}\phi_1 & \text{for } a = 1,\ldots,k\\ \phi_2 & \text{for } a = k+1,\ldots,n.\end{cases} \tag{2.13}$$

In terms of this ansatz in the limit $n \to 0$ the replica summations can be performed according to the following simple rule: $\sum_{a=1}^{n} \phi_a = k\phi_1 + (n-k)\phi_2 \to k(\phi_1 - \phi_2)$. The saddle-point equation (2.12) then turns into two equations for ϕ_1 and ϕ_2:

$$-\tau\phi_{1,2} + \phi_{1,2}^3 = \beta k h_0^2 (\phi_1 - \phi_2). \tag{2.14}$$

Assuming that $\beta k h_0^2 \ll \tau$ (the explanation of this strange assumption—considering that we are interested in the $\beta \to \infty$ limit!—will be given below), in the leading order one gets:

$$\phi_1 \simeq +\sqrt{\tau} \qquad \phi_2 \simeq -\sqrt{\tau}. \tag{2.15}$$

From equation (2.11) one obtains the corresponding energy of the above RSB saddle-point solution:

$$E_k = -\tfrac{1}{2}k\tau(\phi_1^2 - \phi_2^2) + \tfrac{1}{4}k(\phi_1^4 - \phi_2^4) - \tfrac{1}{2}\beta h_0^2 k^2 (\phi_1 - \phi_2)^2 \simeq -2\beta k^2 h_0^2 \tau + \mathrm{O}(h_0^4). \tag{2.16}$$

Now, similarly to the calculations of section 2.1 for the zero-temperature free energy one obtains, summing the contributions from all these saddle points:

$$\begin{aligned} F(h_0) &= -\lim_{n\to 0} \frac{1}{\beta n}(Z_n - 1) \simeq -\lim_{n\to 0} \frac{1}{\beta n}(Z_{\mathrm{RS}} - 1) - \lim_{n\to 0} \frac{1}{\beta n} Z_{\mathrm{RSB}} \\ &= -\lim_{n\to 0} \frac{1}{\beta n}\left[\exp\left(\frac{1}{4}\beta n \tau^2\right) - 1\right] - \frac{1}{\beta}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(2\beta^2 k^2 h_0^2 \tau) \\ &= -\frac{1}{4}\tau^2 - \frac{1}{\beta}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \ln\left[1 + \exp\left(2\beta h_0 x \sqrt{\tau}\right)\right]. \end{aligned} \tag{2.17}$$

Taking the limit $\beta \to \infty$ one finally gets the result:

$$F(h_0) \simeq -\frac{1}{4}\tau^2 - \frac{1}{\beta} 2\beta h_0 \sqrt{\tau} \int_0^{+\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} x \exp\left(-\frac{1}{2}x^2\right) = -\frac{1}{4}\tau^2 - \frac{2h_0\sqrt{\tau}}{\sqrt{2\pi}} \tag{2.18}$$

which coincides with equation (2.10).

It is worth noting that the summation of the series in equation (2.17) can also be performed in the other way (a similar technique has been used by Campellone, who was able to compute in this way the finite N corrections to the free energy of the Random Energy Model in the high temperature phase [3]):

$$F_{\mathrm{RSB}} = -\frac{1}{\beta}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(2\beta^2 k^2 h_0^2 \tau) = \frac{1}{2\mathrm{i}\beta}\int_C \frac{\mathrm{d}z}{z \sin(\pi z)} \exp(2\beta^2 z^2 h_0^2 \tau) \tag{2.19}$$

where the integration goes over the contour in the complex plane shown in figure 1(a). Then we can move the contour to the position shown in figure 1(b), and after the change of integration variable:

$$z \to [2\beta^2 h_0^2 \tau]^{-1/2} \mathrm{i}x \tag{2.20}$$

in the limit $\beta \to \infty$ we have:

$$\sin(\pi z) \simeq \frac{1}{\beta} \mathrm{i}\pi [2h_0^2 \tau]^{-1/2} x. \tag{2.21}$$

Then, taking into account also the contribution from the pole at $x = 0$ for the integral in equation (2.19) we get:

$$F_{\mathrm{RSB}} = \frac{h_0\sqrt{2\tau}}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2}[\exp(-x^2) - 1] = -\frac{2h_0\sqrt{\tau}}{\sqrt{2\pi}} \tag{2.22}$$

which again coincides with results (2.10) and (2.18).

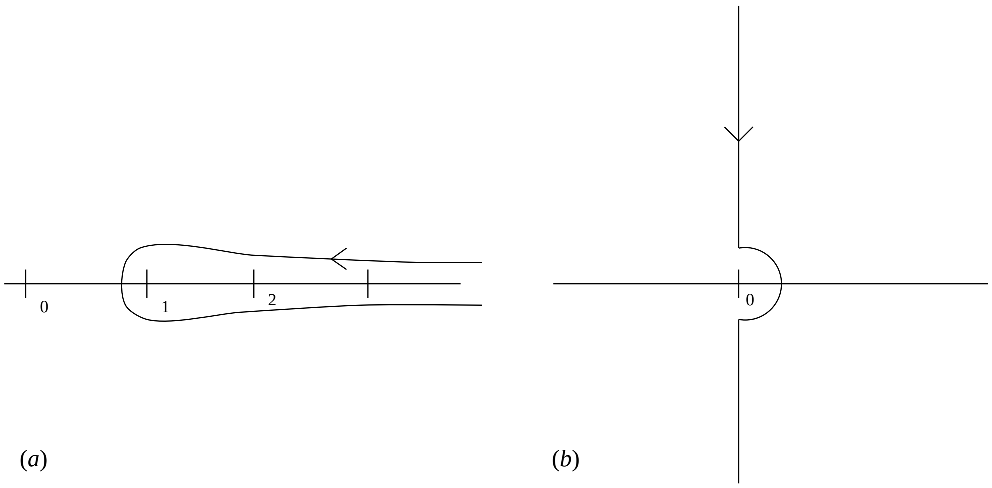

Figure 1. The contours of integration in the complex plane used for summing the series. (*a*) The original contour. (*b*) The deformed contour.

This little exercise with the integral representation of the divergent series in equation (2.19) shows in particular that the 'effective' value of the parameter $\beta k \to \beta z$ which enters into the saddle-point equation (2.14) and scales (according to (2.20)) as $(h_0\sqrt{\tau})^{-1}$. That is why in the zero-temperature limit the 'effective' value of the factor $\beta k h_0^2 \sim h_0/\sqrt{\tau}$ in equation (2.14) can be assumed to be small compared with τ (for small fields $h_0 \ll \tau^{3/2}$).

2.2.1. Replica fluctuations. Because of the non-commutativity of the limits $n \to 0$ and $\beta \to \infty$, one cannot get the exact result by keeping only the saddle-point states of the replica Hamiltonian. Actually, averaging over the quenched disorder involves the effects of sample-to-sample fluctuations which in terms of the replica formalism manifest themselves as the contribution from the replica fluctuations. In other words, to get exact results in terms of replicas the contribution from the saddle points is not enough, and one has to integrate over replica fluctuations even in the zero-temperature limit.

This phenomenon can be easily demonstrated for the above example of the 'soft' Ising model. Let us take into account the contribution from the Gaussian replica fluctuations near the RS saddle point $\phi_a = \phi_{\mathrm{RS}} = \sqrt{\tau}$:

$$\phi_a = \phi_{\mathrm{RS}} + \varphi_a. \tag{2.23}$$

From equation (2.11) for the RS part of the partition function we get:

$$\begin{aligned} Z_{\mathrm{RS}} &= \exp(\tfrac{1}{4}\beta n\tau^2) \int \mathrm{d}\varphi_a \, \exp\left\{ -\beta \sum_{a,b}^{n} (\tau\delta_{ab} - \tfrac{1}{2}\beta h_0^2)\varphi_a\varphi_b \right\} \\ &\simeq \exp\left\{ \frac{1}{4}\beta n\tau^2 + \frac{\beta n h_0^2}{4\tau} - \frac{1}{2} n \ln(\beta\tau) \right\}. \end{aligned} \tag{2.24}$$

Therefore, in the zero-temperature limit one obtains the following contribution to the free energy:

$$F_{\mathrm{RS}} = -\frac{1}{4}\tau^2 - \frac{h_0^2}{4\tau}. \tag{2.25}$$

We see that at $T = 0$ there exists a finite contribution $\sim h_0^2/\tau$ due to the replica fluctuations. In the particular example considered the value of h_0 was assumed to be small, and this contribution can be treated as a small correction. However, we should keep in mind that the contribution from the replica fluctuations in general could appear to be of the same order as that from the saddle points. Therefore, the calculations we are going to perform in the next section for less trivial examples taking into account only saddle-point states cannot pretend to give exact results, giving only the scaling dependence from the parameters of a model.

2.2.2. Saddle points. In the above calculations of the free energy for the 'soft' Ising system we have taken into account only the contribution from the two *minima* of the double-well potential. The existence of the third saddle point, which is the maximum at $\phi = 0$, has been ignored. In this particular example such an algorithm looks natural. However, in less trivial systems very often it is not easy to distinguish the types of the saddle points involved. Moreover, it could be very hard to impose a simple and robust 'discrimination' rule with respect to different types of saddle points, which would not block the calculations at the very start.

Because of this, we would like to propose a somewhat modified scheme of calculations which takes into account *all* saddle points. In the above example of the 'soft' Ising model the third saddle point (the maximum) is at $\phi = 0$. Then, instead of ansatz (2.13) let us represent the replica vector ϕ_a as follows:

$$\phi_a = \begin{cases} +\sqrt{\tau} & \text{for } a = 1, \ldots, k \\ -\sqrt{\tau} & \text{for } a = k+1, \ldots, k+l \\ 0 & \text{for } a = k+l+1, \ldots, n. \end{cases} \tag{2.26}$$

For the corresponding 'energy' (in the limit $n \to 0$) from the replica Hamiltonian (2.11) one easily finds:

$$H_{kl} = -\tfrac{1}{4}\tau^2(k+l) - \tfrac{1}{2}\beta h_0^2 \tau (k-l)^2 + \mathrm{O}(h_0^4). \tag{2.27}$$

Note that in terms of ansatz (2.26) the RS state ($k = l = 0$), $\phi_a = \phi_0 = 0$ has zero energy, so that it gives no contribution to the free energy.

The combinatoric factor in the $n \to 0$ limit is now:

$$\frac{n!}{k!l!(n-k-l)!} \to n \frac{(-1)^{k+l-1}}{k+l} \frac{(k+l)!}{k!l!}. \tag{2.28}$$

Thus, for the free energy (for $\beta \to \infty$) we obtain:

$$F(h_0) = -\frac{1}{\beta} \sum_{k+l=1}^{\infty} \frac{(-1)^{k+l-1}}{k+l} \frac{(k+l)!}{k!l!} \exp\left\{\frac{1}{4}\beta\tau^2(k+l) + \frac{1}{2}\beta^2 h_0^2 \tau (k-l)^2\right\}. \tag{2.29}$$

This series can be summed up in a similar way as the ones in equations (2.8) and (2.17):

$$F(h_0) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} Dx \ln\left[1 + \exp\left\{\frac{\beta}{4}\tau^2 + \beta h_0 \sqrt{\tau} x\right\} + \exp\left\{\frac{\beta}{4}\tau^2 - \beta h_0 \sqrt{\tau} x\right\}\right]. \tag{2.30}$$

In the limit $\beta \to \infty$ one finds:

$$F(h_0) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} Dx \left[\frac{\beta}{4}\tau^2 + \beta h_0 \sqrt{\tau} |x|\right] = -\frac{1}{4}\tau^2 - \frac{2h_0\sqrt{\tau}}{\sqrt{2\pi}}. \tag{2.31}$$

Again, we get the correct result. While it might seem at first sight somewhat 'magic', at least some aspects of this computation can be understood. In the example considered (as well as in the further examples to be studied below) the relevant states, which contribute to the free energy, have negative energy $-E(h)$. Then, in the low-temperature limit the partition function of a given sample is $Z \simeq \exp(+\beta E(h))$. Therefore, in the limit $\beta \to \infty$ the free energy can be represented with exponential accuracy as follows:

$$F(h_0) = -\frac{1}{\beta}\overline{\ln Z} \simeq -\frac{1}{\beta}\overline{\ln(1+Z)} = -\frac{1}{\beta}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\overline{Z^m}. \tag{2.32}$$

One can easily check that after averaging $\overline{Z^m} \equiv Z_m$ and taking into account the contributions of the two minima of the corresponding replica Hamiltonian H_m one recovers the series in equation (2.29).

The only 'magic' rule which should be followed in the direct replica calculations is that the 'background' state, $\phi_0 = 0$ (the one with zero energy) in the ansatz for the replica vector ϕ_a of the type (2.26) should be placed in the *last* group of replicas. Using this rule, the series obtained for the free energy will correspond to the above interpretation (2.32).

2.3. One-well potential

Consider now how the method works in the case where the Hamiltonian has only one minimum:

$$H = \frac{1}{\alpha}\phi^\alpha - h\phi \tag{2.33}$$

where $\phi \geqslant 0$ and $\alpha \geqslant 2$, and the random field h is again described by the Gaussian distribution (2.2). For $\alpha = 4$ this system can be interpreted as the variant of the Hamiltonian (2.9) in the limit of strong magnetic fields.

In the zero-temperature limit the free energy is defined by the ground state $\phi(h) = h^{1/(\alpha-1)}$ for $h > 0$, and $\phi = 0$ for $h \leqslant 0$. Its energy is $E(h) = -\frac{\alpha-1}{\alpha}h^{\alpha/(\alpha-1)}$ for $h > 0$, and $E = 0$ for $h \leqslant 0$. Therefore, for the averaged zero-temperature free energy we find:

$$\begin{aligned} F(h_0) &= -\frac{\alpha-1}{\alpha}\int_0^{+\infty}\frac{\mathrm{d}h}{\sqrt{2\pi h_0^2}}h^{\frac{\alpha}{\alpha-1}}\exp\left\{-\frac{h^2}{2h_0^2}\right\} \\ &= -h_0^{\frac{\alpha}{\alpha-1}} \times \frac{\alpha-1}{\alpha}\int_0^{+\infty}\frac{\mathrm{d}x}{\sqrt{2\pi}}x^{\frac{\alpha}{\alpha-1}}\exp(-\frac{1}{2}x^2) \equiv A(\alpha) \times h_0^{\frac{\alpha}{\alpha-1}}. \end{aligned} \tag{2.34}$$

In terms of replicas, the corresponding replicated Hamiltonian is:

$$H_n = \frac{1}{\alpha}\sum_{a=1}^{n}\phi_a^\alpha - \frac{1}{2}\beta h_0^2\sum_{a,b=1}^{n}\phi_a\phi_b. \tag{2.35}$$

This Hamiltonian has a trivial 'background' extremum at $\phi = 0$ with zero energy. Therefore, following the scheme proposed in the previous section, we look for non-trivial saddle-point solutions in terms of the following ansatz:

$$\phi_a = \begin{cases} \phi & \text{for } a = 1, \ldots, k \\ 0 & \text{for } a = k+1, \ldots, n. \end{cases} \tag{2.36}$$

For the corresponding Hamiltonian and the saddle-point equation (in the limit $n \to 0$) one gets:

$$H_k = \frac{1}{\alpha}k\phi^\alpha - \frac{1}{2}\beta h_0^2 k^2\phi^2 \tag{2.37}$$

$$\phi^{\alpha-1} - \beta h_0^2 k\phi = 0. \tag{2.38}$$

The solution of this equation and the corresponding energy are:

$$\phi = (\beta k h_0^2)^{\frac{1}{\alpha-2}} \tag{2.39}$$

$$H_k = -\frac{1}{\beta}\frac{\alpha-2}{2\alpha}(\beta k)^{\frac{2(\alpha-1)}{\alpha-2}} h_0^{\frac{2\alpha}{\alpha-2}}. \tag{2.40}$$

(Note, that although one can try with more RSB steps in the replica vector ϕ_a it can be easily proved that there exists only one type of the non-trivial solution given by the ansatz (2.36).) Then, in terms of the procedure described above for the free energy we have:

$$F(h_0) = -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\exp\left\{\frac{\alpha-2}{2\alpha}(\beta k)^{\frac{2(\alpha-1)}{\alpha-2}} h_0^{\frac{2\alpha}{\alpha-2}}\right\}. \tag{2.41}$$

The summation of this series can be performed in terms of the integral representation equation (2.19):

$$F(h_0) = \frac{1}{2\mathrm{i}\beta}\int_C \frac{\mathrm{d}z}{z\sin(\pi z)}\exp\left\{\frac{\alpha-2}{2\alpha}(\beta z)^{\frac{2(\alpha-1)}{\alpha-2}} h_0^{\frac{2\alpha}{\alpha-2}}\right\} \tag{2.42}$$

where the integration goes over the contour in the complex plane shown in figure 1(a). Then, again, we move the contour to the position shown in figure 1(b) and redefine the integration variable:

$$z \to \frac{1}{\beta} h_0^{-\frac{\alpha}{\alpha-1}}\mathrm{i}x. \tag{2.43}$$

In the limit $\beta \to \infty$ we have:

$$\sin(\pi z) \simeq \frac{1}{\beta}\mathrm{i}\pi h_0^{-\frac{\alpha}{\alpha-1}} x \tag{2.44}$$

and

$$F(h_0) = -h_0^{\frac{\alpha}{\alpha-1}}\frac{1}{2\pi}\int_{C_1}\frac{\mathrm{d}x}{x^2}\exp\left\{\frac{\alpha-2}{2\alpha}(\mathrm{i}x)^{\frac{2(\alpha-1)}{\alpha-2}}\right\} \equiv B(\alpha)\times h_0^{\frac{\alpha}{\alpha-1}}. \tag{2.45}$$

Thus, we have obtained the correct scaling of the free energy as a function of h_0. Note however, that although it is also possible to calculate the value of the coefficient $B(\alpha)$ in the integral (2.45), such a calculation would not make much sense because to obtain the correct coefficient (which is given by the integral in (2.34)) one would need to take into account replica fluctuations which we have neglected here.

2.4. The toy model

Let us consider now a slightly less trivial example of a zero-dimensional system which cannot be solved by elementary algebra. This system, generally called the 'toy model', consists of a single degree of freedom, ϕ, evolving in an energy landscape which is the sum of a quadratic well and a Brownian potential. The Hamiltonian is:

$$H = \tfrac{1}{2}\mu\phi^2 + V(\phi) \tag{2.46}$$

where $V(\phi)$ is the random potential described by the Gaussian distribution:

$$P[V(\phi)] \sim \exp\left\{-\frac{1}{4g}\int \mathrm{d}\phi\left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2\right\}. \tag{2.47}$$

The V distribution is characterized by its first two moments:

$$\begin{aligned}&\overline{(V(\phi)-V(\phi'))^2}=2g|\phi-\phi'|\\&\overline{V(\phi)}=0\\&\overline{V(\phi)V(\phi')}=C-g|\phi-\phi'|\end{aligned}\tag{2.48}$$

where C is an irrelevant constant. This problem was introduced originally as a toy, zero-dimensional version of the interface in the random field Ising model [4]. It has the virtue of showing explicitly how the most standard field-theoretic methods such as the perturbation theory, iteration methods or supersymmetry get fooled in this problem, in the low μ, low-temperature limit, by the existence of many metastable states [5–8]. The main point is that at low enough temperatures the usual perturbation theory does not work and a qualitatively reasonable theory must involve the effects of the replica symmetry breaking. This has been demonstrated within the replica Gaussian variational approximation [9, 10].

One quantity which one would like to calculate in such a system is the value of $\overline{\langle\phi^2\rangle}$ in the limit of the zero temperature. Using simple energy arguments one can easily estimate what must be the scaling dependence of this quantity on the parameters μ and g. For a given value of ϕ the loss of energy due to the attractive potential in the Hamiltonian (2.46) is $\sim\mu\phi^2$. A possible gain of energy due to the random potential according to statistics (2.47), (2.48) can be estimated as $\sim\sqrt{g}\sqrt{\phi}$. Optimizing the total energy $E\sim\mu\phi^2-\sqrt{g}\sqrt{\phi}$ with respect to ϕ one finds that

$$\overline{\langle\phi^2\rangle}=C_2\frac{g^{2/3}}{\mu^{4/3}}.\tag{2.49}$$

This result tells that the typical energy minimum of the Hamiltonian (2.46) lies at a finite distance from the origin. The scaling (2.49), which is obviously right, is not so easy to derive from some field-theoretic methods which could also be used in higher dimensions, and there is no known exact result for the constant C_2 at the moment.

Let us try to calculate the value of $\overline{\langle\phi^2\rangle}$ in the zero temperature limit using the method considered above. The replicated Hamiltonian of the system (2.46) is:

$$H_n=\tfrac{1}{2}\mu\sum_{a=1}^{n}\phi_a^2+\tfrac{1}{2}\beta g\sum_{a,b=1}^{n}|\phi_a-\phi_b|.\tag{2.50}$$

The corresponding saddle-point equations are:

$$\mu\phi_a+\beta g\sum_{b=1}^{n}\mathrm{Sign}(\phi_a-\phi_b)=0.\tag{2.51}$$

(Note that in this formula, whenever there is some ambiguity, one should always assume that there is at some intermediate step a short-scale regularization. Therefore, one must interpret for instance $\mathrm{Sign}(0)=0$.) Let us first look for non-trivial solutions of equation (2.51). It can be easily proven that within the 'one-step' RSB ansatz (2.13) no non-trivial solutions exist. Let us consider the 'two-steps' ansatz for the replica vector ϕ_a:

$$\phi_a=\begin{cases}\phi_1 & \text{for } a=1,\ldots,k\\ \phi_2 & \text{for } a=k+1,\ldots,k+l\\ \phi_3 & \text{for } a=k+l+1,\ldots,n.\end{cases}\tag{2.52}$$

From equation (2.51) one finds the following equations for $\phi_{1,2,3}$ (in the limit $n\to 0$):

$$\begin{aligned}&\mu\phi_1+\beta gl\ \mathrm{Sign}(\phi_1-\phi_2)-\beta g(k+l)\ \mathrm{Sign}(\phi_1-\phi_3)=0\\&\mu\phi_1+\beta gk\ \mathrm{Sign}(\phi_2-\phi_1)-\beta g(k+l)\ \mathrm{Sign}(\phi_2-\phi_3)=0\\&\mu\phi_3+\beta gk\ \mathrm{Sign}(\phi_3-\phi_1)+\beta gl\ \mathrm{Sign}(\phi_3-\phi_2)=0.\end{aligned}\tag{2.53}$$

The solution of these equations is:

$$\phi_1 = -\frac{g}{\mu}\beta k \qquad \phi_2 = +\frac{g}{\mu}\beta l \qquad \phi_3 = \frac{g}{\mu}\beta(l-k) \tag{2.54}$$

and the corresponding energy is (in the limit $n \to 0$):

$$E_{kl} = -\frac{\beta^2 g^2}{2\mu} kl(k+l). \tag{2.55}$$

It can be proven that there exist no other solutions of the saddle-point equation (2.51) with a number of RSB steps larger than two.

Therefore (after taking the limit $n \to 0$) for the RSB part of the free energy we get the following series (see equation (2.28)):

$$\begin{aligned} F_{\mathrm{RSB}} &= -\frac{1}{\beta n}\sum_{k+l=1}^{n} \frac{n!}{k!l!(n-k-l)!}\exp(-\beta E_{kl}) \\ &\to -\frac{1}{\beta}\sum_{k+l=1}^{\infty} \frac{(-1)^{k+l-1}}{k+l}\frac{(k+l)!}{k!l!}\exp\{\lambda kl(k+l)\} \end{aligned} \tag{2.56}$$

where

$$\lambda = \frac{\beta^3 g^2}{2\mu} \to \infty. \tag{2.57}$$

We again carry the summation of the asymptotic series (2.56) with the integral method mentioned in section 2.2:

$$F_{\mathrm{RSB}} = \frac{1}{\beta(2\mathrm{i})^2}\int\int_C \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1+z_2)\sin(\pi z_1)\sin(\pi z_2)}\frac{\Gamma(z_1+z_2+1)}{\Gamma(z_1+1)\Gamma(z_2+1)}\exp\{\lambda z_1 z_2(z_1+z_2)\} \tag{2.58}$$

where the integrations over $z_{1,2}$ both go around the contour in the complex plane shown in figure 1(*a*).

Shifting the contour of integration to the position shown in figure 1(*b*), and redefining the integration variables: $z_{1,2} \to \lambda^{-1/3}\mathrm{i}x_{1,2}$ in the limit $\beta \to \infty$ ($\lambda^{-1/3} \to 0$) one gets:

$$F_{\mathrm{RSB}} = \frac{1}{\beta}\frac{\lambda^{1/3}}{2\pi^2}\left\{\int\int_0^{+\infty}\mathrm{d}x_1\,\mathrm{d}x_2\left[\frac{\sin(x_1x_2(x_1+x_2))}{x_1x_2(x_1+x_2)} + \frac{\sin(x_1x_2(x_1-x_2))}{x_1x_2(x_1-x_2)}\right]\right\}. \tag{2.59}$$

Substituting the value of $\lambda = \beta^3 g^2/2\mu$ we finally get the result for the zero-temperature free energy:

$$F_{\mathrm{RSB}} = \frac{g^{2/3}}{\mu^{1/3}}\frac{\sqrt{3}\Gamma(\frac{1}{6})}{4\pi^{3/2}}. \tag{2.60}$$

To this piece we must now add the replica-symmetric contribution. The saddle-point equations have the trivial solution: $\phi_a = 0$ with the corresponding energy $E_0 \equiv H_n[\phi_a = 0] = 0$. As we want to get a quantitative result for the constant C_2, we must also include the contribution from the replica fluctuations around this saddle point. This cannot be done just at the level of integrating the quadratic fluctuations. We shall rather make the following (strong) assumption, namely that this whole 'RS' part of the free energy, including the replica fluctuations, is given by the Gaussian replica variational method [10, 8, 9]. We do not have a very convincing argument to support this hypothesis; we just point out that this Gaussian variational method involves the Gaussian integration over replica fields which in a sense is 'symmetric' with respect to the point $\phi_a = 0$. In the end the hypothesis is best supported by the good result one gets for C_2. We denote the Gaussian variational contribution by F_{rv}, and our conjecture is that $F = F_{\mathrm{rv}} + F_{\mathrm{RSB}}$.

According to equation (2.46):

$$\overline{\langle\phi^2\rangle} = 2\frac{\partial F}{\partial\mu} = \overline{\langle\phi^2\rangle}_{\mathrm{rv}} - \frac{g^{2/3}}{\mu^{4/3}}\frac{\Gamma(\frac{1}{6})}{2\sqrt{3}\pi^{3/2}}. \tag{2.61}$$

Using the result of [8] for the value of $\overline{\langle\phi^2\rangle}_{\mathrm{rv}}$ we finally get:

$$\overline{\langle\phi^2\rangle} = \frac{g^{2/3}}{\mu^{4/3}}\left(\frac{3}{(4\pi)^{1/3}} - \frac{\Gamma(\frac{1}{6})}{2\sqrt{3}\pi^{3/2}}\right) \simeq 1.001\,81\frac{g^{2/3}}{\mu^{4/3}}. \tag{2.62}$$

We have compared this result with some numerical simulations of the problem. The scaling in μ and g is obviously correct, the only point to check is the prefactor C_2. Choosing for instance the values of the parameters $\mu = 1$ and $g = 2\sqrt{\pi}$ (when the replica variational method gives $\overline{\langle\phi^2\rangle}_{\mathrm{rv}} = 3$) we obtain from (2.62) the analytical prediction: $\overline{\langle\phi^2\rangle} \simeq 2.3291$. The numerical simulation was done at zero temperature, with the same values of μ and g. The ϕ interval $[-8, 8]$ is discretized in $2N$ points, on which one generates a random potential as in (2.46). The exhaustive scan gives the minimum, from which one computes $\overline{\langle\phi^2\rangle}$. We average over 100 000 samples. The number of points $2N$ ranged from 2^8 to 2^{16}, in this regime there is no systematic N dependance. There is no systematic error due to the finite width of the interval since we have checked that, within our statistics, there is no sample for which the minimum is found with $|\phi| \geqslant 7$. The result of the simulation is $\overline{\langle\phi^2\rangle} \simeq 2.45 \pm 0.02$. The value predicted by our replica saddle-point summation is rather close, although there is a clear small discrepancy.

2.4.1. The result for $\overline{\langle\phi^4\rangle}$. To be sure that this relatively good agreement of our prediction for $\overline{\langle\phi^2\rangle}$ with the numerical result is not just a coincidence we have performed similar calculations for the next-order correlator $\overline{\langle\phi^4\rangle}$. The computations, which are similar to the ones we have just presented but more cumbersome, are given in the appendix. The result is:

$$\overline{\langle\phi^4\rangle} = \overline{\langle\phi^4\rangle}_{\mathrm{rv}} + \overline{\langle\phi^4\rangle}_{\mathrm{RSB}} = \frac{g^{4/3}}{\mu^{8/3}}\left(\frac{27}{(4\pi)^{2/3}} - \frac{17\sqrt{3}[\sin(\pi/12) + \cos(\pi/12)]}{3\sqrt{\pi}\,\Gamma(\frac{1}{6})\sin(\pi/6)}\right). \tag{2.63}$$

For the values of the parameters $\mu = 1$ and $g = 2\sqrt{\pi}$ (when the replica variational method gives $\overline{\langle\phi^4\rangle}_{\mathrm{rv}} = 27$) we obtain: $\overline{\langle\phi^4\rangle} \simeq 16.25$. The numerical result is obtained by using the same procedure as above and gives $\overline{\langle\phi^4\rangle} \simeq 17.05 \pm 0.2$. Again these number are close but there is a significant difference.

Clearly, the vector type of RSB that we have been using on all these zero-dimensional problems is somewhat strange, and we cannot assert that we control all of its aspects (in particular the fact that the replica fluctuations around the RS saddle point are summed by the Gaussian variational method is still unclear). However, in all these cases, and in particular in the non-trivial case of the toy model, we have obtained good results using this simple method. Therefore, we now turn to its application to more elaborate problems, starting with systems in one dimension.

3. Directed polymers in random media

The problem of a directed polymer in a random medium is an important problem which recently has been greatly studied [11]. Although the situation in $1 + 1$ dimensions, with a delta correlated potential, is relatively well understood, there are still a lot of uncertainties about more complicated cases.

We shall consider a one-dimensional case with long-range correlations of the potential. It is described by a one-dimensional scalar field system with the following Hamiltonian:

$$H = \int_0^L \mathrm{d}x \left[\frac{1}{2} \left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \right)^2 + V(x, \phi) \right] \tag{3.1}$$

where the random potentials $V(x, \phi)$ are described by the Gaussian distribution with *non-local* correlations with respect to the fields ϕ:

$$\overline{V(x, \phi) V(x', \phi')} = \delta(x - x')[\text{constant} - g(\phi - \phi')^{2\alpha}] \tag{3.2}$$

where $0 < \alpha < 2$.

This problem naturally arises, with $\alpha = \frac{1}{2}$, when one considers an interface in the two-dimensional random field Ising model at low temperatures: then the field ϕ just describes the lateral fluctuations in the interface, in a solid-on-solid approximation.

One first basic question that we would like to answer concerns the scaling behaviour of the lateral fluctuations. Let the value of the field $\phi(x)$ be put to zero at the origin: $\phi(x = 0) \equiv 0$. Then one would like to know how the average value of the field at $x = L$, $\overline{\langle \phi(L)^2 \rangle}$, scales with L:

$$\overline{\langle \phi(L)^2 \rangle} \equiv \overline{\left(Z^{-1} \int \mathrm{d}\phi_0\, \phi_0^2 \int_{\phi(0)=0}^{\phi(L)=\phi_0} D\phi(x) \exp(-\beta H[\phi(x), V]) \right)} \sim L^{2\zeta} \tag{3.3}$$

where the partition function Z (for a given realization of the random potential) is given by the integration over all the trajectories $\phi(x)$ with only one boundary condition $\phi(x = 0) = 0$. The 'wandering exponent' ζ has been computed in the case of local correlations of the random potential, it is then equal to $\frac{2}{3}$ [12]. In the case of non-local correlations such as (3.2), it is believed that this exponent should be equal to $\frac{3}{2}(2 - \alpha)$ at small enough α. This is the result that is obtained from the Gaussian variational ansatz [10], and it can also be derived from a mapping to the Burgers (or the KPZ) equation and a study of this equation through a dynamical renormalization group procedure [13].

A simple derivation of this scaling can be obtained by an energy balance argument given by Imry and Ma [14]. Let the value of the field be equal to ϕ_0 at $x = L$. Then the loss of energy due to the gradient term in the Hamiltonian (3.1) can be estimated as $E_g \sim \phi_0^2/L$. The gain of energy due to the random potential term, according to equation (3.2), can be estimated as $E_V \sim -\sqrt{L}\sqrt{g}\phi_0^\alpha$. By optimizing E_g and E_V with respect to ϕ_0 one finds:

$$\phi_0 \sim L^{\frac{3}{2(2-\alpha)}} g^{\frac{1}{2(2-\alpha)}}. \tag{3.4}$$

In this section we will demonstrate how this result can be obtained in the zero-temperature limit in terms of the proposed replica saddle-point method. The replicated Hamiltonian is:

$$H_n = \int_0^L \mathrm{d}x \left[\frac{1}{2} \sum_{a=1}^{n} \left(\frac{\mathrm{d}\phi_a(x)}{\mathrm{d}x} \right)^2 + \frac{1}{2}\beta g \sum_{a,b=1}^{n} (\phi_a(x) - \phi_b(x))^{2\alpha} \right]. \tag{3.5}$$

Strictly speaking, the systematic way of solving this problem following our general method is as follows. One must find n saddle-point trajectories $\phi_a(x)$ for fixed n boundary conditions $\phi_a(L)$, then one has to derive the corresponding energy $\tilde{H}_n[\phi_a(L)]$, and finally one has to find the saddle-point solutions with respect to the values of $\phi_a(L)$.

Here we shall follow a much simpler strategy. Since it is obvious that there always exists the trivial solution $\phi(x) \equiv 0$, we will suppose that the correct scaling can be obtained simply by taking into account one non-trivial saddle-point trajectory. In other words, from the very begining we are going to look for the saddle-point solutions within the following ansatz:

$$\phi_a(x) = \begin{cases} \phi(x) & \text{for } a = 1, \ldots, k \\ 0 & \text{for } a = k+1, \ldots, n. \end{cases} \tag{3.6}$$

Comparing this ansatz with the zero-dimensional exercises of the previous section, we see that it should amount to assuming that the lowest lying configuration dominates. This is certainly true since one knows [15, 16] that the metastable states have an excitation energy which scales as L^{ω}, with $\omega = 2\zeta - 1$. Substituting this ansatz into the replica Hamiltonian (3.5) in the limit $n \to 0$ one gets:

$$H_k = k \int_0^L \mathrm{d}x \left[\frac{1}{2} \left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \right)^2 - \beta k g \phi^{2\alpha}(x) \right]. \tag{3.7}$$

As usual (see the previous section) the free energy is defined by the series:

$$F(L) \sim -\frac{1}{\beta} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(-\beta H_k) \tag{3.8}$$

where the value H_k is defined by the corresponding saddle-point solution for $\phi(x)$.

The saddle-point trajectory is defined by the following differential equation:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -2\alpha\beta k g \phi^{2\alpha - 1} \tag{3.9}$$

with the boundary conditions: $\phi(0) = 0$ and $\phi(L) = \phi_0$. This equation can be easily integrated:

$$\int_0^{\phi(x)} \frac{\mathrm{d}\phi}{\sqrt{\lambda - \phi^{2\alpha}}} = x\sqrt{2\beta k g} \tag{3.10}$$

where the integration constant λ is defined by the boundary condition:

$$\int_0^{\phi_0} \frac{\mathrm{d}\phi}{\sqrt{\lambda - \phi^{2\alpha}}} = L\sqrt{2\beta k g}. \tag{3.11}$$

Substituting this solution into the Hamiltonian (3.7), we obtain after some simple algebra:

$$H_k = k\left[-\beta k g \lambda L + \sqrt{2\beta k g} \int_0^{\phi_0} \mathrm{d}\phi \sqrt{\lambda - \phi} \right]. \tag{3.12}$$

Taking the derivative of H_k with respect to ϕ_0 (and taking into account the constraint (3.11)) one finds the following saddle-point solution:

$$\phi_0 \sim L^{\frac{1}{1-\alpha}} (\beta k g)^{\frac{1}{2(1-\alpha)}} \tag{3.13}$$

and $\lambda = \phi_0^{2\alpha}$. Its energy (3.12) is:

$$H_k = -\frac{(\text{constant})}{\beta} (\beta k)^{\frac{2-\alpha}{1-\alpha}} L^{\frac{1+\alpha}{1-\alpha}} g^{\frac{1}{1-\alpha}}. \tag{3.14}$$

Now we proceed as before, introducing an integral representation of the series (3.8) and a rescaling of the integration variable by a factor of $\frac{1}{\beta} L^{-\frac{1+\alpha}{2-\alpha}} g^{-\frac{1}{2-\alpha}}$. Then we get the scaling of the free energy:

$$F(L) \sim L^{\frac{1+\alpha}{2-\alpha}} g^{\frac{1}{2-\alpha}} \tag{3.15}$$

from which we obtain the scaling of ϕ_0 as a function of L:

$$\phi_0(L) \sim L^{\frac{3}{2(2-\alpha)}} g^{\frac{1}{2(2-\alpha)}} \tag{3.16}$$

which coincides with result (3.4) given by the naive energy arguments, as well as by more elaborate calculations.

Although the example demonstrated in this section provides no new results we hope that the proposed method could turn out to also be useful when applied for directed polymers with smaller α, or in larger dimensions.

4. Random field Ising model in D dimensions

Since the topic of the random field Ising model covers an enormous amount of literature (see e.g. [17]), it would be rather difficult to give any brief introductory review. Here, however, we are mainly concerned with how the method we have proposed works in various situations. Therefore, we will concentrate only on one particular aspect of the problem.

It is well known that the main problem in the studies of the low-temperature phase in the random field Ising model is that one has to perform the summation over numerous local minima states, which seems to be impossible to do within the framework of the usual perturbation theory [17]. It has been proposed recently that, because of these local minima states a special 'intermediate' (separating paramagnetic and ferromagnetic phase) spin-glass-like thermodynamic state could set in around the critical point, and moreover, this state is characterized by a replica symmetry breaking in the corresponding correlation functions [18]. At low temperatures, and when the width of the distribution of the random field is not too small, the same phenomenon must be present. It was proposed long ago [19], and later elaborated in [20], that the metastable states in this regime should be characterized by some 'instanton in replica space'. Our method provides one more step in the elaboration of this idea.

We consider the random field Ising model in terms of the usual Ginzburg–Landau Hamiltonian in D dimensions:

$$H = \int \mathrm{d}^D x \, [\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\tau\phi^2 + \tfrac{1}{4}g\phi^4 - h(x)\phi] \tag{4.1}$$

where the random fields $h(x)$ are described by the δ-correlated Gaussian distribution:

$$P[h(x)] = \prod_x \left[\frac{1}{\sqrt{2\pi h_0^2}} \exp\left(-\frac{h^2(x)}{2h_0^2} \right) \right]. \tag{4.2}$$

The corresponding replica Hamiltonian is:

$$H_n = \int \mathrm{d}^D x \left[\tfrac{1}{2} \sum_{a=1}^{n} (\nabla\phi_a)^2 + \tfrac{1}{2}\tau \sum_{a=1}^{n} \phi_a^2 + \tfrac{1}{4} g \sum_{a=1}^{n} \phi_a^4 - \tfrac{1}{2} h_0^2 \sum_{a,b=1}^{n} \phi_a \phi_b \right]. \tag{4.3}$$

According to the procedure developed in previous sections we are going to look for the most simple non-trivial saddle-point solutions at the background of the trivial one, $\phi_a(x) \equiv 0$. In terms of the ansatz:

$$\phi_a(x) = \begin{cases} \phi(x) & \text{for } a = 1, \dots, k \\ 0 & \text{for } a = k+1, \dots, n. \end{cases} \tag{4.4}$$

The replica Hamiltonian (4.3) reads in the $n \to 0$ limit as:

$$H_k = k \int \mathrm{d}^D x \, [\tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}(h_0^2 k - \tau)\phi^2 + \tfrac{1}{4}g\phi^4]. \tag{4.5}$$

Consider for simplicity the situation at $\tau = 0$ (notice that here we work close to the critical temperature. The use of our saddle-point technique allows us to study the system at tree level, which is supposed to give the leading singularities close to T_c [21]). The corresponding saddle-point equation is:

$$-\Delta\phi - \lambda\phi + g\phi^3 = 0 \tag{4.6}$$

where $\lambda = h_0^2 k$. As usual, the free energy is given by the series:

$$F(h_0) \sim -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(-H_k) \tag{4.7}$$

where the value of H_k is defined by the corresponding saddle-point solution of equation (4.6).

At this stage we see that the situation is getting rather different from the ones studied in the previous sections. If we choose the obvious space-independent solution $\phi = (\lambda/g)^{1/2}$, we would find that the value of H_k is proportional to the volume V of the system: $H_k = -\frac{1}{4}k(\lambda^2/g)V = -\frac{1}{4g}k^3 h_0^4 V$. Then, the summation of the series (4.7) would immediately yield a free energy proportional to $V^{1/3}$ and not to V. Therefore this solution, as well as any other solution with an energy H_k proportional to the volume of the system, is irrelevant for the bulk properties.

Thus, we have to look for *localized* solutions: the ones which are local in space (breaking translation invariance) and which have *finite* energy. Let us first assume that such an 'instanton'-type solution exists (see below), and that for a given k it is characterized by the spatial size $R(k)$. Then, if we take into account only one-instanton contribution (or in other words we consider a gas of *non-interacting* instantons), due to the trivial entropy factor V/R^D (this is the number of positions of the object of the size R in the volume V) we get a free energy proportional to the volume:

$$F(h_0) \sim -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{V}{R^D} \exp(-H_k) \tag{4.8}$$

where H_k must be finite (volume independent).

It is easy to understand that equation (4.6) indeed has localized solutions. Let us assume that the value of the field $\phi(x)$ is such that $\lambda\phi^2 \gg g\phi^4$. Then in a first approximation the saddle-point equation (4.6) is linear:

$$\Delta\phi + \lambda\phi = 0. \tag{4.9}$$

The simplest possible spherically symmetric solutions of this equation in D dimensions are the well known Bessel-type functions. In particular oscillating solutions exist which have a finite value $\phi(r=0) \equiv \phi_0$ at the origin and which decay to zero at $r \to \infty$ (like $\sim r^{-(D-1)/2} \sin r$). For example, in dimension $D = 3$ this solution is simply:

$$\phi(r) = \phi_0 \frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}}. \tag{4.10}$$

In dimensions D these solutions have a finite spatial scale:

$$R(k) = \lambda^{-1/2} = (h_0^2 k)^{-1/2} \tag{4.11}$$

and finite energy:

$$H_k = -(\text{constant})k\phi_0^2\lambda^{-\frac{D-2}{2}}. \tag{4.12}$$

At the level of equation (4.9) itself, the value of ϕ_0 remains arbitrary (the equation is linear). On the other hand, from the point of view of the energy this is not an extremum since the energy explicitly depends on the value of ϕ_0 (this is the saddle-point solution for the fixed boundary condition $\phi(r = 0) = \phi_0$). If we let the value of ϕ_0 be free in the absence of the nonlinear term $g\phi^4$ it would, of course, fall down to infinity. However, if we take into account the term $g\phi^4$ in the 'exact' Hamiltonian (4.5) it is natural to expect that ϕ_0 will stabilize around the saddle-point value

$$\phi_0^2 = \frac{\lambda}{g}. \tag{4.13}$$

The above qualitative arguments can be easily verified for the model double-well potential: $\tilde{U}(\phi) = -\frac{1}{2}\phi^2$ for $|\phi| \leqslant \sqrt{\lambda/g}$ and $\tilde{U}(\phi) = +\infty$ for $|\phi| > \sqrt{\lambda/g}$, taken instead of the 'real' one: $U(\phi) = -\frac{1}{2}\phi^2 + \frac{1}{4}g\phi^4$. In this case for any $|\phi_0| \leqslant \sqrt{\lambda/g}$ there exists the exact Bessel-like saddle-point solution with finite energy (4.12), and real extremum of the Hamiltonian would be achieved at $\phi_0 = \pm\sqrt{\lambda/g}$.

Let us calculate the contribution of such solutions to the free energy. Substituting into the series (4.8) the energy of the solution (4.12), the estimate for the value of ϕ_0 (4.13) and the characteristic size of the solution (4.11), together with $\lambda = h_0^2 k$ we get:

$$F(h_0) \sim -V \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (h_0^2 k)^{\frac{D}{2}} \exp\left[\frac{(\text{constant})}{g h_0^2} (h_0^2 k)^{\frac{6-D}{2}}\right]. \tag{4.14}$$

We see that the series is getting strongly divergent only at dimensions $D < 6$. This is the only regime where the considered saddle-point solutions provide a relevant contribution.

Now, following the scheme developed in the previous sections, we turn to the integral representation and rescale the integration variable by a factor $(gh_0^2)^{\frac{2}{6-D}} h_0^{-2}$, which gives a free energy with the following scaling in the limit $gh_0^2 \ll 1$:

$$\frac{F(h_0)}{V} \sim \frac{1}{g}(gh_0^2)^{\frac{4}{6-D}}. \tag{4.15}$$

Besides, using the same scaling $k \sim (gh_0^2)^{\frac{2}{6-D}} h_0^{-2}$ for the characteristic spatial scale of the saddle-point solutions (4.11), which could be interpreted as a kind of disorder induced *finite* correlation length (near $T = T_c$, as we shall see in more detail below), we obtain:

$$R_c(h_0) \sim (gh_0^2)^{-\frac{1}{6-D}}. \tag{4.16}$$

In the same way one gets the estimate for the value of the 'disorder parameter' $\overline{\phi^2} \sim \phi_0^2 \simeq \frac{1}{g}(h_0^2 k)$:

$$\phi_0^2 \sim \frac{1}{g}(gh_0^2)^{\frac{2}{6-D}}. \tag{4.17}$$

Finally, one can easily obtain the estimate for the value of the temperature interval τ_c around T_c where all the above qualitative calculations make sense. Formally the derivation of the saddle-point solutions has been done for $\tau = 0$. Actually, according to the replica Hamiltonian (4.5) the calculations should remain correct until $|\tau| \ll h_0^2 k$. Using the above scale estimate for k one finds the upper bound for the value of τ:

$$|\tau| \ll \tau_c \sim (gh_0^2)^{\frac{2}{6-D}}. \tag{4.18}$$

This value of τ_c can be interpreted as the estimate for the temperature interval around T_c where the supposed disorder dominated (spin-glass-type) phase sets in.

Of course, the procedure proposed in this section is still incomplete. In a self-consistent approach one should study the effects produced by the interactions between these instanton solutions, not talking about the effects of the critical fluctuations. At the present stage we are not able to say anything about the ferromagnetic phase transition itself and in particular about the behaviour of the corresponding ferromagnetic order parameter.

Nevertheless, we shall now show that these simple replica instanton estimates are quite reasonable and can in fact be recovered in terms of (completely independent) simple scaling arguments. Indeed, let us come back to the original random field Hamiltonian (4.1). Configurations of the field $\phi(x)$ which correspond to local minima satisfy the saddle-point equation:

$$-\Delta\phi(x) + \tau\phi(x) + g\phi^3(x) = h(x). \tag{4.19}$$

Let us estimate at which spatial and temperature scales the random fields give a dominant contribution. We consider a large region Ω_L of linear size $L \gg 1$. The spatially averaged value of the random field in this region is:

$$h(\Omega_L) \equiv \frac{1}{L^D}\int_{x\in\Omega_L} \mathrm{d}^D x\, h(x). \tag{4.20}$$

Correspondingly, the typical average value of the random field in this region of size L is:

$$h_L \equiv [\overline{h^2(\Omega_L)}]^{1/2} = h_0 L^{-D/2}. \tag{4.21}$$

Then the estimate for the typical value of the order parameter field ϕ_L in this region can be obtained from the saddle-point equation:

$$\tau\phi_L + g\phi_L^3 = h_L. \tag{4.22}$$

Then, as long as:

$$\tau\phi_L \ll g\phi_L^3 \tag{4.23}$$

the typical value of ϕ_L inside such clusters is dominated by the random field:

$$\phi_L \sim \left(\frac{h_L}{g}\right)^{1/3} \sim \left(\frac{h_0}{g}\right)^{1/3} L^{-D/6}. \tag{4.24}$$

Now let us estimate up to which characteristic size of the cluster the external fields can dominate. According to (4.23) and (4.24) one gets:

$$L \ll \frac{(gh_0^2)^{1/D}}{\tau^{3/D}}. \tag{4.25}$$

On the other hand, the estimation of the order parameter in terms of the equilibrium equation (4.22) can be correct only on length scales much larger than the size of the fluctuation region which is equal to the correlation length (of the pure system) $R_c \sim \tau^{-\nu}$. Thus, one has the lower bound for L:

$$L \gg \tau^{-\nu}. \tag{4.26}$$

Therefore, the region of parameters where the external fields dominate is:

$$\tau^{-\nu} \ll \frac{(gh_0^2)^{1/D}}{\tau^{3/D}} \tag{4.27}$$

or

$$\tau^{3-\nu D} \ll gh_0^2. \tag{4.28}$$

Such a region of temperatures near T_c exists only if:

$$\nu D < 3. \tag{4.29}$$

In this case the temperature interval near T_c in which the order parameter configurations are mainly defined by the random fields is:

$$\tau_c(h_0) \sim (gh_0^2)^{\frac{1}{3-\nu D}}. \tag{4.30}$$

In the mean-field theory (which correctly describes the phase transition in the pure system for $D > 4$) $\nu = \frac{1}{2}$. Thus, according to condition (4.29) the above non-trivial temperature interval τ_c exists only in dimensions $D < 6$. Substituting $\nu = \frac{1}{2}$ into (4.30) we get:

$$\tau_c(h_0) \sim (gh_0^2)^{\frac{2}{6-D}}. \tag{4.31}$$

Then, the random field defined spatial scale can be estimated from (4.25):

$$L_c(h_0) \sim (gh_0^2)^{-\frac{1}{6-D}}. \tag{4.32}$$

Correspondingly, the typical value of the order parameter field at scales $L_c(h_0)$ is obtained from equation (4.24):

$$\phi_{L_c}^2 \sim \frac{1}{g}(gh_0^2)^{\frac{2}{6-D}}. \tag{4.33}$$

The energy density is estimated as $\frac{E}{V} \sim \phi_{L_c} h_{L_c}$. Taking into account (4.21) and (4.33) we find:

$$\frac{E}{V} \sim \frac{1}{g}(gh_0^2)^{\frac{4}{6-D}}. \tag{4.34}$$

We see that we get, through these simple arguments, a region around T_c where the disorder induces a finite correlation length. Furthermore, the estimates for $\frac{E}{V}$, L_c, ϕ_{L_c} and τ_c perfectly coincide with the results obtained in terms of our previous replica saddle-point method, equations (4.15)–(4.18). Both approaches clearly hold only in a regime where critical fluctuations can be neglected.

5. Conclusions

We have proposed a method to analyse random systems by summing up various saddle-point contributions in the replicated Hamiltonian. We think that it may open a new route in this type of study. In particular, the application to finite-dimensional systems, which we started here with the directed polymer on the one hand, and with the random field Ising model on the other hand, looks quite interesting. Indeed we have seen in this last case how this method allows us to take into account instanton contributions which are usually out of reach of most analytic methods in these systems. Such instanton contributions have been argued to be important for a long time [2, 3]. We think we can get them under control with the present approach.

Clearly our method is still not totally understood in all details. We have pointed out that it involves one single basic rule, stating the way one has to order the various saddle points in replica space. Within this hypothesis it gives reasonable results in all the cases we have checked so far, but of course more studies are needed to justify this hypothesis.

Appendix: Computation of the fourth moment in the toy model

Using the saddle point solution (2.54) we have:

$$\overline{\langle\phi^4\rangle}_{\rm RSB} = \sum_{k+l=1}^{n} \frac{n!}{k!l!(n-k-l)!}[k\phi_1^4 + l\phi_2^4 + (n-k-l)\phi_0^4]\exp\{-\beta E_{kl}\}$$
$$\to \left(\frac{\beta g}{\mu}\right)^4 \sum_{k+l=1}^{\infty} \frac{(-1)^{k+l-1}}{k+l}\frac{(k+l)!}{k!l!} kl(k+l)(3k^2+3l^2-5kl)\exp\{\lambda kl(k+l)\}. \tag{A.1}$$

Proceeding similarly to the calculations of the free energy $F_{\rm RSB}$ (2.58), (2.59) we get:

$$\overline{\langle\phi^4\rangle}_{\rm RSB} = \left(\frac{\beta g}{\mu}\right)^4 \frac{\partial}{\partial\lambda}\{-\frac{1}{(2i)^2}\int\int_C \frac{{\rm d}z_1{\rm d}z_2}{(z_1+z_2)\sin(\pi z_1)\sin(\pi z_2)} \frac{\Gamma(z_1+z_2+1)}{\Gamma(z_1+1)\Gamma(z_2+1)}$$
$$\times(3z_1^2+3z_2^2-5z_1z_2)\exp[\lambda z_1z_2(z_1+z_2)]\}. \tag{A.2}$$

Shifting contour to the position in figure 1(*b*) and redefining $z_{1,2} \to \lambda^{-1/3}{\rm i}x_{1,2}$ in the limit $\beta\to\infty$ ($\lambda^{-1/3}\to 0$) we find:

$$\overline{\langle\phi^4\rangle}_{\rm RSB} = \left(\frac{\beta g}{\mu}\right)^4 \frac{\partial}{\partial\lambda}\left\{-\frac{\lambda^{-1/3}}{2\pi^2}\int\int_{C_1}\frac{{\rm d}x_1{\rm d}x_2}{(x_1+x_2)x_1x_2}(3x_1^2+x_2^2-5x_1x_2)\right.$$
$$\left.\times\exp[-{\rm i}\lambda x_1x_2(x_1+x_2)]\right\}. \tag{A.3}$$

Taking into account the contribution from the pole at $x_{1,2}=0$ after somewhat painful algebra we finally obtain the following result:

$$\overline{\langle\phi^4\rangle}_{\rm RSB} = -\frac{g^{4/3}}{\mu^{8/3}}\frac{17\sqrt{3}[\sin(\pi/12)+\cos(\pi/12)]}{3\sqrt{\pi}\,\Gamma(\frac{1}{6})\sin(\pi/6)}. \tag{A.4}$$

Taking into account the contribution from the replica fluctuations [8]:

$$\overline{\langle\phi^4\rangle}_{\rm rv} = \frac{g^{4/3}}{\mu^{8/3}}\frac{27}{(4\pi)^{2/3}} \tag{A.5}$$

for the fourth-order correlator we get the final result:

$$\overline{\langle\phi^4\rangle} = \overline{\langle\phi^4\rangle}_{\rm rv} + \overline{\langle\phi^4\rangle}_{\rm RSB} = \frac{g^{4/3}}{\mu^{8/3}}\frac{27}{(4\pi)^{2/3}} - \frac{g^{4/3}}{\mu^{8/3}}\frac{17\sqrt{3}[\sin(\pi/12)+\cos(\pi/12)]}{3\sqrt{\pi}\,\Gamma(\frac{1}{6})\sin(\pi/6)}. \tag{A.6}$$

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