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The Large Scale Energy Landscape of Randomly Pinned Objects

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Abstract. — We discuss the large scale effective potential for elastic objects (manifolds) in the presence of a random pinning potential, from the point of view of the functional renormalisation group (FRG) and of the replica method. Both approaches suggest that the energy landscape at large scales is a succession of parabolic wells of random depth, matching on singular points where the effective force is discontinuous. These parabolas are themselves subdivided into smaller parabolas, corresponding to the motion of smaller length scales, in a hierarchical manner. Consequences for the *dynamics* of these pinned objects are underlined.

1. Introduction

The physics of elastic objects pinned by random impurities is certainly one of the most topical current themes of statistical mechanics. The problem is of fundamental importance both from a theoretical point of view (many of the specific difficulties common to disordered systems are at stake) and for applications: the pinning of flux lines in superconductors [1–3], of dislocations, of domain walls in magnets, or of charge density waves [4, 5], controls in a crucial way the properties of these materials. Interestingly, this problem is also intimately connected to surface [6] and crack growth [7] and to turbulence [8].

Two different general approaches have been proposed to describe the *statics* of these pinned manifolds, for which perturbation theory badly fails. The first one is the "functional renormalisation group" (FRG) which aims at constructing the correlation function for the effective pinning potential acting on long wavelengths using renormalisation group (RG) ideas [9, 10]. The second is the variational replica method which combines a Gaussian trial Hamiltonian with "replica symmetry breaking" (RSB) to obtain results in the low temperature, strongly pinned phase [11–13]. Although many of the results of these two approaches actually turn out to be similar [11, 13–17], the feeling that the link between them is missing is rather widespread,

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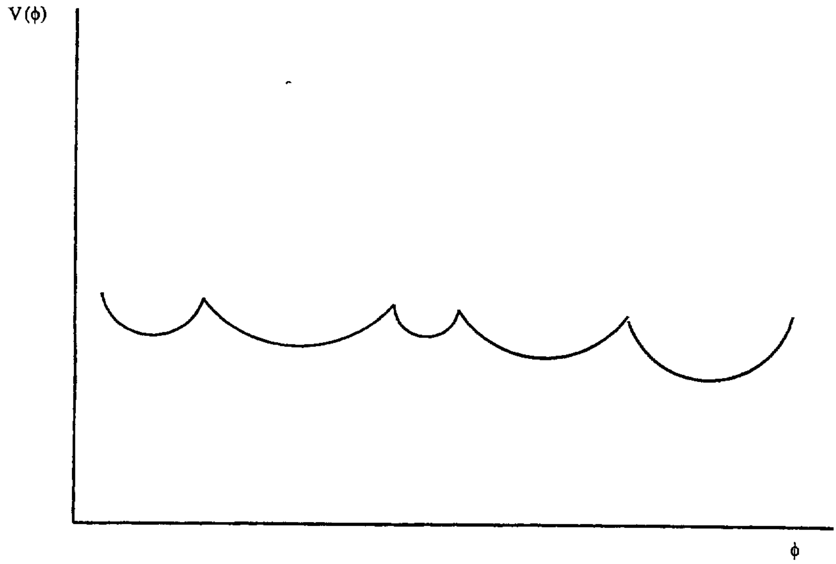

Fig. 1. — Schematic view of the effective energy landscape as a succession of parabolic wells matching at singular point. This picture actually corresponds to a "one-step" replica symmetry breaking scheme.

reflecting the fact that our present general understanding of disordered system is still incomplete.

The aim of this article is to unveil precise connections between these two (sometimes presented as conflicting [10, 16, 18]) theories. We show that both formalisms are indeed struggling to describe an awkward reality: the effective, long wavelength pinning potential has the shape drawn in Figure 1. It is a succession of parabolic wells of random depth, matching on singular points where the effective force (*i.e.* the derivative of the potential) is discontinuous. These discontinuities induce a singularity in the effective potential correlation function, and are encoded in the replica language by the RSB. The replica calculation furthermore provides an explicit construction of this effective (random) potential, and hence, in turn, information on the statistics of – say – the depth of the potential minima. The replica calculation might also shed light on the domain of validity of the FRG, by making more explicit the assumptions on which the latter relies.

Apart from the satisfying possibility of reconciling two rather different microscopic methods, we believe that our construction is very useful to understand the *dynamics* of such objects. For example, their relaxation can be analyzed in terms of hops between the different minima ("traps"), corresponding to metastable long wavelength configurations. The statistics of barrier heights control the trapping time distribution, and hence the low frequency response and its possible aging behaviour [19, 20]. Another interesting situation is the zero temperature depinning transition induced by an external driving field, which has recently been investigated, again using RG ideas for expanding around a mean-field limit [21–23]. However, the results depend on the form of the pinning potential in this mean-field limit. The correct form was surmised by Narayan and Fisher [22] to be the "scalloped" potential of Figure 1. Our calculation, to some extent, confirms their intuition.

The model we consider is the (by now standard) Hamiltonian describing pinned elastic manifolds:

$$\mathcal{H}(\{\vec{\phi}(\mathbf{x})\}) = \int \mathrm{d}^D\mathbf{x} \left[\frac{c}{2} \left(\frac{\partial \vec{\phi}(\mathbf{x})}{\partial \mathbf{x}} \right)^2 + V_0(\mathbf{x}, \vec{\phi}(\mathbf{x})) \right], \tag{1}$$

where $\mathbf{x}$ is a D-dimensional vector labelling the internal coordinates of the object, and $\vec{\phi}(\mathbf{x})$ an N-dimensional vector giving the position in physical space of the point labelled $\mathbf{x}$. Various values of D and N actually correspond to interesting physical situations. For example, $D = 3$, $N = 2$ describes the elastic deformation of a vortex lattice (after a suitable anisotropic generalization of Eq. (1)), $D = 2$, $N = 1$ describes the problem of domain walls pinned by impurities in 3 dimensional space, while $D = 1$ corresponds to the well-known directed polymer (or single flux line) in a $N + 1$ dimensional space. The elastic modulus c measures the difficulty of distorting the structure, and $V_0(\mathbf{x}, \vec{\phi}(\mathbf{x}))$ is a random pinning potential, which we shall choose to be Gaussian with a short range connected correlation function:

$$\overline{V_0(\mathbf{x}, \vec{\phi}) V_0(\mathbf{x}', \vec{\phi}')}_c = NW\delta^D(\mathbf{x} - \mathbf{x}') R_0 \left(\frac{(\vec{\phi} - \vec{\phi}')^2}{N} \right), \tag{2}$$

where W measures the strength of the pinning potential. The scaling with N is chosen to ensure a correct $N \to \infty$ limit, in the sense that the pinning part of the free-energy is of order N, while each component of the pinning force $-\frac{\partial V_0}{\partial \vec{\phi}}$ remains of order 1. In the following, we shall choose for convenience $R_0(y) = \exp(-\frac{y}{2\Delta^2})$ where Δ is the correlation length of the random potential.

One aim of the theory is to understand how the microscopic pinning potential will affect the elastic manifold on long length scales, relevant for macroscopic measurements. In other words, one would like to construct the *effective* pinning potential seen by a low wavevector mode of the structure, after thermalizing the modes with shorter length scales. Both the FRG and the replica approach propose an approximate construction of this effective potential which we now discuss and relate.

2. The Functional Renormalisation Group

In the spirit of the momentum shell renormalisation group, the FRG method consists in writing down a recursion relation for the correlation function of the potential acting on "slow" modes $\vec{\phi}_<$, after "fast" modes $\vec{\phi}_>$ (corresponding to wavevectors in the high-momentum shell $[\Lambda/b, \Lambda]$) have been integrated out using perturbation theory. This procedure has been addressed in considerable detail in reference [10], we present only a brief description of the calculations. At zero temperature the renormalized Hamiltonian is defined by $H_{\mathrm{R}}[\vec{\phi}_<] = \int_{\mathbf{x}} \frac{1}{2}|\nabla\vec{\phi}_<|^2 + \mathcal{V}_{\mathrm{R}}[\vec{\phi}_<]$ and

$$\mathcal{V}_{\mathrm{R}}[\vec{\phi}_<] = \min_{\vec{\phi}_>} \int_{\mathbf{x}} \left\{ \frac{1}{2}|\nabla\vec{\phi}_>|^2 + V(\vec{\phi}_< + \vec{\phi}_>, \mathbf{x}) \right\}, \tag{3}$$

where the original field $\vec{\phi} = \vec{\phi}_< + \vec{\phi}_>$ has been split into low ($\vec{\phi}_<$) and high ($\vec{\phi}_>$) momentum components. The renormalized Hamiltonian H_{R} thus describes the long-distance physics of modes with momenta $k < \Lambda/b$, where Λ is the original short-scale cutoff, and the rescaling factor $b > 1$. The FRG proceeds to determine the minimum in equation (3) perturbatively in $\vec{\phi}_>$. The extreme condition may be expanded in $\vec{\phi}_>$ as

$$\begin{aligned} -\nabla^2\phi^i_> &= -\partial_i V(\vec{\phi}_< + \vec{\phi}_>, \mathbf{x}) \\ &\approx -\partial_i V(\vec{\phi}_<, \mathbf{x}) - \partial_i\partial_j V(\vec{\phi}_<, \mathbf{x})\phi^j_>. \end{aligned} \tag{4}$$

where $\partial_i \equiv \frac{\partial}{\partial \phi^i}$. Defining the Fourier transform $\tilde{V}_{\mathbf{k}}^{ij\cdots} = \partial_i\partial_j \cdot \int_{\mathbf{x}} V(\vec{\phi}_<, \mathbf{x})\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}$, the approximate solution is

$$\Lambda^2 \phi^i_{>,\mathbf{k}} \approx -\tilde{V}^i_{\mathbf{k}} + \Lambda^{-2} \int_{\mathbf{k}'} \tilde{V}^{ij}_{\mathbf{k}-\mathbf{k}'} \tilde{V}^j_{\mathbf{k}'}. \tag{5}$$

Inserting this solution into the energy (Eq. (3)) gives

$$\mathcal{V}_{\mathrm{R}} = \tilde{V}_0 - \frac{1}{2\Lambda^2} \int_{\mathbf{k}}^{>} \tilde{V}^i_{\mathbf{k}} \tilde{V}^i_{-\mathbf{k}} + \frac{1}{2\Lambda^4} \int_{\mathbf{pp}'}^{>} \tilde{V}^{ij}_{\mathbf{k}+\mathbf{k}'} \tilde{V}^i_{-\mathbf{k}} \tilde{V}^j_{-\mathbf{k}'}. \tag{6}$$

where $\int^>$ is restricted to the high-momentum shell. If $\vec{\phi}_<(\mathbf{x})$ is constant over regions of a certain size ℓ, this can be rewritten as an integral of a *local* potential, up to small errors of order $1/\ell$: $\mathcal{V}_{\mathrm{R}}(\vec{\phi}_<) \simeq \int \mathrm{d}\mathbf{x} V_{\mathrm{R}}(\vec{\phi}_<, \mathbf{x})$. Thus, in the long wavelength limit, the renormalized Hamiltonian is well-described simply by a renormalized potential. Its connected correlations can be calculated from the expression

$$\overline{V_{\mathrm{R}}(\vec{\phi}, \mathbf{x}) V_{\mathrm{R}}(\vec{\phi}', \mathbf{x}')}_{\mathrm{C}} = R_{\mathrm{R}} \left(\frac{[\vec{\phi} - \vec{\phi}']^2}{N} \right) \delta(\mathbf{x} - \mathbf{x}'). \tag{7}$$

Assuming that the statistics of the effective potential remains Gaussian, one finds within this first order perturbation theory:

$$R_{\mathrm{R}}(y) = R(y) + \frac{\mathrm{d}l}{8\pi^2} \left[\frac{1}{2} \partial_i\partial_j R \partial_i\partial_j R - \partial_i\partial_j R \partial_i\partial_j R(0) \right] \tag{8}$$

in $D = 4$, where $b = \mathrm{e}^{\mathrm{d}l}$ and $\mathrm{d}l$ is infinitesimal.

Equation (8) is the final result of the mode elimination. The search for fixed points requires the additional step of a rescaling transformation, which restores the original value of the cutoff Λ. Performing this rescaling *via* $\mathbf{x} \to b\mathbf{x}$ and $\phi \to b^\zeta \phi$ results in the full RG equation for the correlator

$$\partial_l R = (\epsilon - 4\zeta) R + \zeta \phi^i \partial_i R + \frac{1}{8\pi^2} \left[\frac{1}{2} \partial_i\partial_j R \partial_i\partial_j R - \partial_i\partial_j R \partial_i\partial_j R(0) \right]. \tag{9}$$

Iteration of this equation from the "initial" condition $R(y) = R_0(y)$ converges towards the *fixed point* $R^*(y)$, describing the long wavelength properties, which has the singular small y expansion [10]

$$R^*(y) - R^*(0) = \epsilon y [a_1 - a_{3/2}\sqrt{y}] + \cdots, \tag{10}$$

where $\epsilon = 4 - D$ is the small parameter justifying the use of perturbation theory. Another way of stating this result is in term of the effective *force* f acting on the manifold, defined as minus the derivative of the effective potential with respect to ϕ. The force correlation function then behaves as

$$\overline{[f^*(\phi) - f^*(\phi')]^2} = 12\epsilon a_{3/2} |\phi - \phi'|. \tag{11}$$

Together with the assumption of Gaussian statistics, this suggests that the effective force acting on the manifold behaves, for $N = 1$, as a *random walk* in ϕ space. This picture was advocated in [10], and was actually used to argue that the next correction in ϵ to would be of order $\epsilon^{3/2}$

3. The Replica Approach

The replica approach is, in some sense, more ambitious, since it provides an explicit probabilistic construction of the effective disordered potential seen by the manifold. On the other hand, the method can only be controlled in the $N \to \infty$ limit, where a Gaussian variational Hamiltonian becomes exact [24]. Let us however stress right away that a Gaussian Hamiltonian in replica space *does not* mean that the actual effective potential which we wish to characterize has Gaussian statistics. As we shall indeed show below, this is not at all the case.

Let us sketch first how the correlation function $R(y)$ can be calculated with replicas and compared with the FRG. (More details can be found in [8, 11, 12]). The average free-energy $F = -\frac{1}{\beta}\overline{\ln Z} \equiv -\frac{1}{\beta}\overline{\ln \int \mathcal{D}\phi \exp[-\beta\mathcal{H}]}$ is computed as usual as the "zero replica" limit $\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n}$. The average of Z^n can be seen as the partition function of the following n-replica Hamiltonian:

$$\mathcal{H}_n = \frac{c}{2}\sum_{a=1}^{n}\int \mathrm{d}^D\mathbf{x}\left(\frac{\mathrm{d}\vec{\phi}^a}{\mathrm{d}\mathbf{x}}\right)^2 - \frac{WN}{2}\sum_{a,b}\int \mathrm{d}^D\mathbf{x}\exp\left[-\frac{(\vec{\phi}^a(\mathbf{x}) - \vec{\phi}^b(\mathbf{x}))^2}{2N\Delta^2}\right], \tag{12}$$

where an effective attraction between replicas has emerged from the disorder average. The idea is to treat this interaction using a trial Hamiltonian for which analytical progress is possible [26]:

$$\mathcal{H}_v \equiv \frac{1}{2}\sum_{a,b}\sum_{\mathbf{k}} \vec{\varphi}^a(-\mathbf{k})G^{-1}_{ab}(\mathbf{k})\vec{\varphi}^b(\mathbf{k}), \tag{13}$$

where $\varphi^a(\mathbf{k}) \equiv L^{-\frac{D}{2}}\int \mathrm{d}^D\mathbf{x}\ \phi^a(\mathbf{x})\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}$, and L is the "linear" size of the manifold.

The trial free-energy obtained with $\mathcal{H}_v$ depends on G_{ab} and reads $\mathcal{F}_v[G] = \langle\mathcal{H}_n\rangle_v - \frac{N}{2\beta}\mathrm{Tr}\ln G$; the optimal matrix G is then determined by minimizing $\mathcal{F}_v[G]$, which leads to a set of self-consistent equations for G_{ab}. The point now is that the structure of G_{ab} in replica space can be non trivial in the limit $n \to 0$, corresponding to "replica symmetry breaking". The physical meaning of this procedure has already been described in detail in [8, 11, 27], and we shall come back to it later. Before describing the solution to these self-consistent equations in the regime $D \lesssim 4$, one should clarify first in what sense the replica calculation allows one to characterize the large scale pinning potential. Since the trial Hamiltonian is factorized over Fourier modes, one can isolate a particular, very slow mode $\mathbf{k}_0 \to 0$. The effective force acting on $\vec{\varphi}_0 \equiv \vec{\varphi}(\mathbf{k}_0)$ is $f^\mu_\Omega(\vec{\varphi}_0) = -\frac{1}{\beta}\frac{\partial}{\partial\varphi^\mu_0}\ln\mathcal{P}_\Omega(\vec{\varphi}_0)$, where $\mathcal{P}_\Omega(\vec{\varphi}_0)$ is the probability to observe $\vec{\varphi}_0$ for a given realization of the random pinning potential Ω. It is thus clear that in order to compute, say, the correlation function of $\vec{f}$, one should study the object:

$$\overline{f^\mu_\Omega(\vec{\varphi}_0)f^\nu_\Omega(\vec{\varphi}'_0)} = \lim_{n\to 0}\frac{4}{n^2}\frac{\partial^2}{\partial\varphi^\mu_0\partial\varphi'^\nu_0}\overline{[\mathcal{P}_\Omega(\vec{\varphi}_0)]^{n/2}[\mathcal{P}_\Omega(\vec{\varphi}'_0)]^{n/2}}. \tag{14}$$

The last quantity is directly calculable, since the Gaussian ansatz asserts that

$$\overline{\prod_{a=1}^{n}\mathcal{P}_\Omega(\vec{\varphi}^a_0)} = \sum_{\pi}\exp[-\frac{\beta}{2}\vec{\varphi}_0^{\pi(a)}G^{-1}_{ab}(\mathbf{k}_0)\vec{\varphi}_0^{\pi(b)}], \tag{15}$$

where G is the optimal matrix determined *via* the self-consistent equations and π denotes all the permutations of the replica indices. (All the saddle points only differing by permutation of the indices must be taken into account). The quantity in the right hand side of equation (14) corresponds to the choice $\varphi^a_0 = \varphi_0$ for $n/2$ indices, and $\varphi^a_0 = \varphi'_0$ for the other $n/2$. The next trick to compute (15) is to notice that in this case one can write $\varphi_0^a \equiv \frac{1}{2}\left[\varphi_0(1+\sigma_a)+\varphi_0'(1-\sigma_a)\right]$, where $\sigma_a = \pm 1$ are fictitious Ising spins which pick up a particular permutation, provided $\sum_{a=1}^{n} \sigma_a = 0$. The technique for working out the sums over such spin configurations has been developed in the appendix D of reference [8]. Within a Parisi ansatz for the matrix G, the final result for the force correlation, written in the case of $N = 1$ to keep notations simple, is the following:

$$\overline{f_\Omega(\varphi_0) f_\Omega(\varphi_0')} = \frac{2}{\beta} c k_0^2 - \frac{4}{\beta^2} \frac{\partial^2}{\partial\varphi_0 \partial\varphi_0'} \int_0^\infty \mathrm{d}h \Psi(h, u=0) \tag{16}$$

where $\Psi(h,u)$ satisfies a non linear partial differential equation:

$$-\frac{\partial\Psi}{\partial u} = \frac{1}{2}\frac{\mathrm{d}q}{\mathrm{d}u}\left(\frac{\partial^2\Psi}{\partial h^2} + u[\frac{\partial\Psi}{\partial h}]^2\right), \tag{17}$$

where $0 \leq u \leq 1$ is the Parisi variable, indexing the pairs of replica indices $a \neq b$ in the limit $n \to 0$. The function $q(u)$ is related to the matrix $G(\mathbf{k}, u)$ through:

$$q(u) = -\beta \frac{(\varphi_0 - \varphi_0')^2}{4} G^{-1}(\mathbf{k}_0, u) \tag{18}$$

and the boundary condition is

$$\Psi(h, u=1) = \ln\left(1 + \exp[-2h - 2q(1) + 2\int_0^1 \mathrm{d}u\; q(u)]\right) \tag{19}$$

Hence, once $G(\mathbf{k}, u)$ is determined, the correlation function of the effective potential acting on mode $\mathbf{k}_0$ is determined by solving (17), which depends on $\varphi_0 - \varphi_0'$ through $q(u)$.

The solution of the self-consistent equations for $G(\mathbf{k}, u)$ was discussed in [11, 12]. Let us specialize to the case $N = \infty$, and introduce two important physical quantities, namely:

- The Larkin-Ovchinnikov length ξ_{LO} separating a "weakly distorted" regime for $|\mathbf{x}| < \xi_{\mathrm{LO}}$, where all the displacements induced by the random potential are small compared to the correlation length of the potential Δ, from a strongly distorted regime. Simple dimensional arguments lead to [12, 28]

$$\xi_{\mathrm{LO}} \equiv \left(\frac{c^2\Delta^4}{\hat{W}}\right)^{\frac{1}{4-D}}, \tag{20}$$

where $\hat{W}$ is a rescaled potential strength, defined as $\hat{W} = (2\pi)^2 W/(4-D)$. The reason for introducing this rescaling in $\frac{1}{4-D}$ comes from the non trivial phase diagram around dimension $D = 4$ [29]. Indeed, it is easily seen from the study of the linearised, random force problem, that a "weak disorder" regime with non trivial wandering exponent only exists when $\hat{W}$ is small enough. If one keeps the original W fixed and lets the dimension D go to 4, one enters a different phase (which actually survives for $D > 4$) [29].

- A "Reynolds" number Re (this terminology comes from the analogy with Burgers' equation [8]), defined as the ratio of the elastic energy stored in a volume ξ_{LO}^D to the temperature $1/\beta$. We shall define Re as

$$\mathrm{Re} \equiv \beta \mathcal{C}(D) \hat{W}^{\frac{2-D}{4-D}} (c\Delta^2)^{\frac{D}{4-D}}, \tag{21}$$

where $\mathcal{C}(D)$ is a dimension dependent number. Note that for $D = 4 - \epsilon$ with ϵ small, $\mathcal{C} = \epsilon^2/2$.

We shall only consider the case of low temperature and weak disorder, so that $\mathrm{Re} \gg 1$ and $\xi_{\mathrm{LO}} \gg a$, where $a = 2\pi/\Lambda$ is the small scale lattice constant which regularizes the integrals over $\mathbf{k}$. Under these conditions, we obtain the following result for $D > 2$:

$$G^{-1}(\mathbf{k}, u) = -\sigma_1 \left(\frac{u}{u_c}\right)^{\frac{4-D}{D-2}} \quad \text{for} \quad u \le u_c \tag{22}$$

$$= -\sigma_1 \quad \text{for} \quad u_c \le u \le 1 \tag{23}$$

with $\sigma_1 = \beta \hat{W} \epsilon / \Delta^2$ and $u_c = 1/\mathrm{Re}$.

The non trivial dependence of $G^{-1}(\mathbf{k}, u)$ on u corresponds to continuous "replica symmetry breaking". Let us now analyze the partial differential equation (17) in the limit $\beta \to \infty$. To this aim, we introduce the notation $\gamma = \frac{6-2D}{D-2}$ and the following rescaled variables:

$$u = u_c v \qquad \psi = u_c \Psi \qquad g = \beta \sigma_1 u_c^2 \left(\frac{4-D}{8(D-2)}\right) (\vec{\varphi}_0 - \vec{\varphi}_0')^2 \qquad z = u_c h / \sqrt{g}. \tag{24}$$

Equation (17) then transforms into

$$\frac{\partial \psi}{\partial v} = -v^{\gamma} [\psi'' + v \psi'^2], \tag{25}$$

(the $'$ means $\frac{\partial}{\partial z}$), with boundary condition (in the limit $\beta \to \infty$):

$$\psi(z, v = 1) = -2\left((D-2)g + z\sqrt{g}\right) \Theta\left(-(D-2)g - z\sqrt{g}\right), \tag{26}$$

where Θ is the step function. The correlation of free energies (16) thus involves, after change of variables, the integral

$$\mathcal{I} \equiv \int_0^{\infty} \mathrm{d}h \Psi(h, u = 0) = -\frac{\sqrt{g}}{u_c^2} \int_0^{\infty} \mathrm{d}z \; z \psi'(z, v = 0). \tag{27}$$

Under this form, the problem of evaluating $\mathcal{I}$ for small $|\vec{\varphi}_0 - \vec{\varphi}_0'|$ can simply be treated by solving equation (25) for $\psi'(z, v)$, perturbatively in g. The result reads – see Appendix:

$$\mathcal{I} = \frac{1}{u_c^2} \left[\frac{g}{\gamma + 1} - \mathcal{L}(\gamma) g^{3/2}\right], \tag{28}$$

with $\mathcal{L}(\gamma)$ a complicated function of γ. In the limit $D = 4 - \epsilon$, $\gamma \simeq -1 + \frac{\epsilon}{2}$, and $\mathcal{L}(\gamma) \simeq 2\sqrt{\pi}$. Transforming back to the original variables, we find, in the limit $k_0 \to 0$:

$$R_{\mathrm{RSB}}(y) - R_{\mathrm{RSB}}(0) = -\epsilon \hat{W} \frac{y}{2\Delta^2} \left(1 - \frac{\sqrt{\pi}}{2} \frac{\sqrt{y}}{\Delta \xi_{\mathrm{LO}}^{D/2}}\right), \tag{29}$$

with $y = (\vec{\varphi}_0 - \vec{\varphi}_0')^2$. Quite remarkably, equation (29) has the same form as the FRG result, equation (10), provided $\hat{W}$ is chosen in such a way that ξ_{LO} remains fixed as $D \to 4$. This $y^{3/2}$ behaviour was first obtained within a replica theory in [8] in the case $D = 1$ (corresponding to Burgers' turbulence), where the solution has a simpler, "one-step" structure (valid for $D < 2$): $G^{-1}(\mathbf{k}, u) = -\sigma_1 \Theta(u - u_c)$.

4. Physical Interpretation

4.1. Shocks and Relationship with the Burgers' Equation. — As mentioned above, the Gaussian variational ansatz does not mean that the statistics of V_{Ω}^* is Gaussian. Let us first discuss the replica construction of the effective potential in the simpler case $D = 1$ where a one step solution holds [8, 11]. In this case, one has:

$$V^*_\Omega(\varphi) = -\frac{1}{\beta} \ln \left[\sum_\alpha \mathrm{e}^{-\beta F_\alpha - [(\varphi-\varphi_\alpha)^2/(u_c \Delta^2)]} \right], \tag{30}$$

where α label the "states", centered around φ_α and of free-energy F_α, both depending on the "sample" Ω. The major prediction of the replica theory is that the F_α are exponentially distributed for "deep" states [30], *i.e.*:

$$\rho(F_\alpha) \propto_{F_\alpha \to -\infty} \exp(-\beta u_c |F_\alpha|). \tag{31}$$

The full distribution of the effective force $\frac{\partial V^*_\Omega}{\partial \vec{\varphi}_0}$ (corresponding to the velocity in the Burgers problem) was analyzed in detail in [8]. Using the turbulence language, it was found that the velocity field organizes in a "froth-like" structure of $N-1$ dimensional shocks of vanishing width in the limit $\mathrm{Re} \to \infty$. Correspondingly, the potential has for $N = 1$ the shape drawn in Figure 1: it is made of parabolas matching at angular points – the shocks. The singular behaviour of the force-force correlation function, equation (11), is due to the fact that with a probability proportional to the "distance" $|\vec{\varphi}_0 - \vec{\varphi}'_0|$, there is a shock which gives a *finite* contribution to $\vec{f}(\vec{\varphi}_0) - \vec{f}(\vec{\varphi}'_0)$. This means in particular that all the moments $\overline{|\vec{f}(\vec{\varphi}_0) - \vec{f}(\vec{\varphi}'_0)|^p}$ grow as $|\vec{\varphi}_0 - \vec{\varphi}'_0|$ for $p \geq 1$, instead of $|\vec{\varphi}_0 - \vec{\varphi}'_0|^{p/2}$ as for Gaussian statistics. It is not clear how this strong departure from Gaussian statistics can be incorporated in an FRG treatment (see Sect. 4.3 below).

The relation with Burgers' equation is not coincidental and actually quite interesting. Keeping $N = 1$ for simplicity, consider a toy model for the FRG mode elimination in which the renormalized effective potential is defined as

$$\beta V_{\mathrm{R}}(\varphi_<) = -\ln \left[\int \mathrm{d}\varphi_> \mathrm{e}^{-\beta[(c\Lambda^2/2)\varphi_>^2 + V_0(\varphi_< + \varphi_>)]} \right]. \tag{32}$$

This means that $V_{\mathrm{R}}(\varphi_<)$ is precisely the Cole-Hopf solution of the Burgers equation [32]:

$$\frac{\partial V(\varphi,t)}{\partial t} = \frac{1}{2\beta c\Lambda^2} \frac{\partial^2 V(\varphi,t)}{\partial \varphi^2} - \frac{c\Lambda^2}{2} \left(\frac{\partial V(\varphi,t)}{\partial \varphi} \right)^2 \tag{33}$$

with

$$V(\varphi, t=0) = V_0(\varphi) \qquad V_{\mathrm{R}}(\varphi) = V(\varphi, t=1). \tag{34}$$

As is well known [32, 33], a random set of initial conditions (here the bare pinning potential acting on φ) develops shocks which separates as time grows, between which the "potential" $V(\varphi)$ has a parabolic shape. Elimination of fast modes in a disordered system thus naturally generates a "scalloped" potential, with singular points (which are smoothed out at finite temperature or finite Re) separating potential wells – the famous "states" appearing in the replica theory. Quite remarkably, this structure was anticipated in [34, 20] using different arguments.

4.2. Full RSB and Multiscale Effective Potential. — In the case of continuous RSB, the effective potential is recursively constructed *via* a set of "Matrioshka doll" Gaussians. It is schematically drawn in Figure 2 for the transverse fluctuations $\phi(\ell) - \phi(0)$. For each length scale ℓ, one can define a characteristic value of the parameter $u(\ell)$ which plays the role of u_c in equation (31) and sets the scale of the energy fluctuations. $u(\ell)$ is such that the diagonal

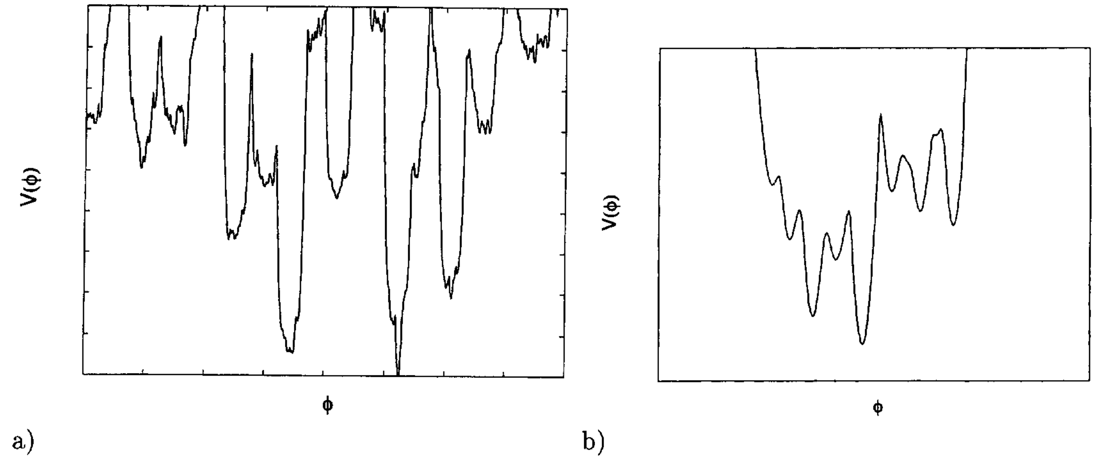

Fig. 2. — a) Multiscale energy landscape corresponding to a full replica symmetry breaking scheme. In this case, the construction is that of parabolas within parabolas, in a hierarchical manner. The depth of the wells (and thus also the height of the barriers) typically grows as $|\phi - \phi'|^{\theta/\zeta}$. The figure actually corresponds to a two-step breaking scheme, with $u_1 = 0.5$ and $u_0 = 0.05$. b) A zoom on a particular region, showing the first level of Gaussians.

part of $G^{-1}(k_0 = 2\pi/\ell)$, namely ck_0^2, is equal to the off diagonal part $G^{-1}(k_0, u)$, which gives $u(\ell) \propto \frac{1}{\beta}\left(\xi_{\rm LO}/\ell\right)^{\theta}$ ($\theta = D-2$ is the "energy" exponent in the case $N = \infty$, and is related to the small u power-law behaviour of $G^{-1}(k_0, u)$). The large scale structure of the effective potential is thus a succession of parabolas of depth $\propto \ell^{\theta}$, but this envelope structure is decorated by hierarchically imbedded parabolas corresponding to all the smaller length scales, between ℓ and $\xi_{\rm LO}$, beyond which the shocks disappear, since one enters into the effectively replica symmetric random force regime. The important point however is that small scale shocks are much more numerous than large scale ones and completely dominate the small y behaviour of $R_{\rm RSB}(y)$: see Figure 2. This explains why the above result (29) is independent of k_0 and only reflects the structure of $G^{-1}(k, u)$ in the vicinity of u_c, corresponding to $k \simeq 1/\xi_{\rm LO}$. On the other hand, quantities like $\overline{[\phi(\ell) - \phi(0)]^2}$ are dominated by the region where $u \simeq u(k_0 = 2\pi/\ell)$, corresponding to large scale moves. More precisely, the main contribution to $\overline{[\phi(\ell)) - \phi(0)]^2}$ comes from minima separated by a distance ℓ^{ζ} which happen to be separated by an energy gap smaller than the temperature [11,17]. This occurs with probability $\propto \beta^{-1} \times (\beta u(k_0))$ (see Eq. (31)).

In other words, the effective potential calculated within the FRG procedure involves an extra step which we have not performed within the replica construction, which is a coarse graining of the ϕ variables. In the FRG calculation, one restricts to configurations which are such that ϕ is constant on scales ℓ, and scales as ℓ^{ζ} [35]. The correct choice of ζ then ensures that there are only a few shocks on the scale ℓ. As we now discuss in a rather conjectural way, this is perhaps why the FRG can still be controlled, the departure from Gaussian statistics being in some sense "weak".

4.3. The FRG in the Presence of Shocks. — To understand the emergence of shocks in the FRG picture, and to assess their impact on the perturbative procedure, it is useful to study the above toy model for the renormalization group, defined by equation (32), which amounts to discarding the internal degrees of freedom. Following reference [10], we write equation (32) at zero temperature (and after a rescaling) as:

$$\mathcal{V}_{\mathrm{R}}(\phi_<) = \min_{\phi_>} \left\{ \frac{1}{2}|\phi_>|^2 + V(\phi_< + \phi_>) \right\}. \tag{35}$$

The validity of the perturbative minimization scheme was discussed in detail in reference [10], *assuming* Gaussian statistics for the random potential V. Errors occur in the perturbative minimization scheme due to an incorrect choice among multiple minima in the effective Hamiltonian for $\phi_>$. For a Gaussian potential, there is an extremely dense set of such minima, and such an error occurs essentially with probability one. The FRG appears to be saved, however, because the *magnitude* of the resulting error in the energy is small (*i.e.* higher order in ϵ).

A rather different picture emerges if one assumes a smooth potential with shocks (*i.e.* slope discontinuities in V) spaced by $O(1)$ distances. To understand the limitations of the perturbative minimization scheme in this case, consider the extreme condition of the toy model,

$$\phi_> = -V'(\phi_< + \phi_>). \tag{36}$$

In a scalloped (piecewise quadratic) potential, a perturbative solution in $\phi_>$ converges to the minimal energy in the local well containing $\phi_> = 0$. For $|V|$ small, this is indeed the global minimum, *unless* a shock occurs within a distance $|\phi_{\mathrm{shock}}| < O(|V|)$, as can be seen by examining the effective Hamiltonian for $\phi_>$ in the neighborhood of a cusp. Provided that a shock is present, however, the incorrect minima is chosen with a probability of $O(1)$, leading to a large error in $\mathcal{V}_{\mathrm{R}}$. Thus for the scalloped potential, instead of persistent small errors, the perturbative minimization scheme is typically correct, but suffers from catastrophic rare events that generate large errors with small probability.

An interesting simplification occurs if one considers a periodic random potential V. Such periodic potentials occur in models of pinned charge density waves [4,22] and random anisotropy XY magnets [5]. It is straightforward to show that repeated applications of the toy model iteration drive the potential towards a form with a single symmetric cusp per period [37]. For such a symmetric form, the perturbative minimization scheme *always* converges to the correct (deepest) minima of the effective potential, *i.e.* the local minimum is always the global minimum. Within the toy model, then, the perturbative minimization scheme appears to be *asymptotically exact*. Although errors may accrue in early stages of the renormalization, these decrease as the length scale grows and the final fixed point form is exact – provided the perturbation theory is carried out to all orders, of course! That the FRG and replica methods lead to essentially the same results in this case was underlined in [13].

The FRG consists, as does any renormalization group, of two parts: the mode elimination (accomplished *via* the perturbative minimization scheme) and the rescaling transformation. The toy model allows a detailed study, in a somewhat schematic way, of the former. Within this framework, the non-analyticity of R emerges in a natural way via the generation of Burgers' shocks. The toy model, however, completely neglects the internal degrees of freedom of the manifold, whose rescaling is crucial for the power-counting in the full FRG. In particular, this rescaling not only leads to the existence of a fixed point for $R(\phi)$, but also formally renders the higher cumulants of V strongly irrelevant.

There appears to be a degree of competition between the mode elimination, which favors shocks and the corresponding highly non-Gaussian distribution for V, and the coarse graining and rescaling transformation, which tends to keep the density of shocks to a low value (at least for small ϵ). A complete description, which is unfortunately not available to us at present, should properly balance these effects against one another. The special considerations applicable for the periodic potential discussed above suggest that the FRG may indeed be well-controlled in that case. More generally, the full accommodation of shocks into the FRG remains a challenging open problem.

4.4. THE 1 + 1 DIRECTED POLYMER. — An explicit model where this construction actually does not require the use of replicas or of the FRG is the $N = 1$, $D = 1$ (Directed Polymer) case. From independent arguments [6,36], one knows that the effective potential $V_x(\phi)$ acting on the "head" of an infinitely long polymer ($x \to \infty$) is a "random walk" in ϕ space: $\overline{[V_x(\phi) - V_x(\phi')]^2} \propto |\phi - \phi'|$. (Notice the difference with equation (11), which concerns the *force*, and not the potential). In particular, there are no shocks in $V_x(\phi)$. Shocks appear when one coarse-grains the description on a scale δ. Let us define a coarse-grained potential on an infinitesimal scale η as

$$V_x^\eta(\rho) \equiv -\frac{1}{\beta} \ln \int_{-\infty}^{\infty} \mathrm{d}\phi \quad \mathcal{K}\left(\frac{(\phi-\rho)^2}{2\eta}\right) \mathrm{e}^{-\beta V_x(\phi)}, \tag{37}$$

where $\mathcal{K}$ is an arbitrary local "filter". Iterating this procedure a large number of times δ/η produces an effective potential V_x^δ which, again, satisfies a Burgers equation, but now with a long range correlated "initial condition" $V_x(\phi)$. As is well known [32,33], shocks also appear in this case, with an average spacing growing as $\delta^{2/3}$. The distribution of distance d between shocks furthermore diverges for small d as $d^{-1/2}$ [33], indicating that there are shocks on all scales smaller than $\delta^{2/3}$. All these results can alternatively be obtained within the replica framework [25,31].

5. Discussion and Perspectives

We have shown in this paper that the FRG and RSB techniques are not contradictory but complementary. They both suggest quite an appealing physical picture: the phase-space of the system is, on large length scales, divided into "cells" corresponding to favourable configurations where the potential is locally parabolic, and whose depth is exponentially distributed. These cells are themselves subdivided into smaller cells, corresponding to larger length scales, etc. This hierarchical construction is similar to the one usually advocated for the phase space of spin-glasses [38], based on Parisi's RSB solution of the Sherrington-Kirkpatrick model [11,20]. The enormous advantage of random manifolds is that this construction can be directly performed in physical space.

An important consequence of this construction is that it allows us to discuss the dynamical properties for finite N [39]. In the case of a one-step RSB, one can directly calculate from equation (30), the distribution of the height of the *barriers* ΔE between two neighbouring wells, and finds that it decays exponentially as $\exp(-\beta u_c \Delta E)$. It is interesting to notice that the barriers thus behave in the same way as energy depths [41], a point recently studied in detail for randomly pinned lines in [42]. A natural picture for the dynamics is thus to imagine that the manifolds jumps from well to well, each of which representing a long-lived conformation of the manifold. Such a picture is corroborated by recent numerical simulations in $D = 1$, $N = 1$ [43]. The lifetime of each "trap" is activated $\tau \simeq \tau_0 \exp(\beta \Delta E)$, and is thus distributed as a power-law $\tau^{-1-u(k)}$ for large τ, where the exponent $u(k) \propto k^\theta$ depends on the "size" of the jump (*i.e.* the mode involved in the change of conformation), small $u(k)$ corresponding to large wavelengths. Then, as emphasized in [19] where precisely the same "trap" picture was advocated for spin-glasses, the dynamics becomes non stationary and aging effects appear at low temperatures and/or long-wavelengths such that $u(k) < 1$. For example, the response of the manifold to a spatially modulated external field is expected to behave, for $t \ll t_w$, as $(t/t_w)^{1-u(k)}$, where t_w is the time elapsed since the quench from high temperature. Correspondingly, the a.c. response should behave, for $\omega t_w \gg 1$, as $(\omega t_w)^{u(k)-1}$, again much in the same way as observed in spin-glasses [19]. For finite N however, one may expect that the exponential distribution of deep states ceases to be valid outside the scaling region, *i.e.* for $\Delta E \ggg \frac{1}{\beta u(k)}$ [19,31]. This will lead to "interrupted aging" for modes such that $\ln t_w \ggg u(k)^{-1}$ These equilibrated modes thereafter only contribute to the stationary part of the response (or correlation).

It is thus rather satisfactory that the "traps" appear naturally in the context of pinned manifolds through the replica description, and that this picture actually complement the "droplet" construction. It would of course be gratifying to understand precisely how these ideas could be extended to finite dimensional spin-glasses.

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Appendix

Perturbation Expansion for Equation (27)

We provide here some intermediate steps of the computation of free energy correlations with the replica method. We need to solve equation (25) with the boundary condition (26), and compute then the integral $\mathcal{I}$ defined in (27). The limit of interest is g small. We work with the derivative $\chi(z,v) = \psi'(z,v)$ which satisfies the equation:

$$\frac{\partial \chi}{\partial v} = -v^{\gamma}[\chi'' + 2v\chi\chi'] \tag{A.1}$$

together with the boundary condition:

$$\chi(z, v=1) = -2\sqrt{g}\Theta\left(-(D-2)g - z\sqrt{g}\right) \ . \tag{A.2}$$

The solution to this differential equation to order g can be written as:

$$\chi(z, v=0) = \chi_1(z) + \chi_2(z) \tag{A.3}$$

where

$$\chi_1(z) = -2\sqrt{g}\int_{-\infty}^{-(D-2)\sqrt{g}} \frac{\mathrm{d}z'}{\sqrt{\dfrac{4\pi}{\gamma+1}}} \exp -\left(\frac{(z-z')^2}{\dfrac{4}{\gamma+1}}\right) \tag{A.4}$$

and

$$\begin{aligned} \chi_1(z) &= 4g\int_0^1 v^{\gamma+1}\mathrm{d}v \\ &\times \int_{-\infty}^{\infty} \frac{\mathrm{d}z'}{\sqrt{\dfrac{4\pi v^{\gamma+1}}{\gamma+1}}} \exp -\left(\frac{(z-z')^2}{\dfrac{4v^{\gamma+1}}{\gamma+1}}\right) \frac{\partial}{\partial z'}\left(\int_{-\infty}^{0} \frac{\mathrm{d}z_1}{\sqrt{\dfrac{4\pi(1-v^{\gamma+1})}{\gamma+1}}} \exp -\left(\frac{(z-z')^2}{\dfrac{4(1-v^{\gamma+1})}{\gamma+1}}\right)\right)^2 \end{aligned} \tag{A.5}$$

Introducing the notation

$$\mathcal{M}_0(x) = \int_x^\infty \frac{\mathrm{d}u}{\sqrt{2\pi}} \mathrm{e}^{-u^2/2} \tag{A.6}$$

we find, after multiplication of χ by z and integration:

$$\begin{aligned} \mathcal{I} \;=\; & \frac{1}{u_c^2}\Bigg(\frac{g}{\gamma+1} \\ & - \sqrt{\frac{2}{\gamma+1}} g^{3/2} [\sqrt{\frac{2}{\pi}}(D-2) - 4\int_0^1 v^{\gamma+1} \mathrm{d}v \int_{-\infty}^{\infty} \mathrm{d}x \mathcal{M}_0(-\frac{x}{v^{\frac{\gamma+1}{2}}}) \mathcal{M}_0^2(\frac{x}{\sqrt{1-v^{\gamma+1}}})] + \ldots\Bigg) \end{aligned} \tag{A.7}$$

Expansion of the last integral for $\epsilon = 4-D$ small. with $\gamma = -1+\epsilon/2$, reveals that the coefficient of $g^{3/2}$ which to leading order should be $\propto \epsilon^{-1/2}$ in fact vanishes, the next term being of order ϵ^0.

References

[1] Fisher D.S., Fisher M.P.A. and Huse D.A., *Phys. Rev. B* **43** (1991) 130.

[2] Blatter G., Feigel'man M.V., Geshkenbin V.B., Larkin A.I. and Vinokur V.M., *Rev. Mod. Phys.* **66** (1994) 4.

[3] for a recent review, see e.g. Kardar M. and Ertas D., in "Scale Invariance, Interfaces and Non-equilibrium dynamics", NATO-ASI held at the Newton Institute (June 1994); to appear, Kardar M., "Lectures on directed paths in random media", Les Houches Summer School on Fluctuating geometries in Statistical Mechanics and Field Theory, in press.

[4] Fukuyama H. and Lee P.A., *Phys. Rev. B* **17** (1978) 535.

[5] Fisher D.S., *Phys. Rev. B* **31** (1985) 7233.

[6] Halpin-Healey T. and Zhang Y.C., *Phys. Rep.* **254** (1995) 217.

[7] Bouchaud J.P., Bouchaud E., Lapasset G. and Planes J., *Phys. Rev. Lett.* **71** (1993) 2240; See also Schmittbuhl J., Roux S., Villotte J.-P. and Maloy K.J., *Phys. Rev. Lett.* **74** (1995) 1787.

[8] Bouchaud J.P., Mézard M. and Parisi G., *Phys. Rev. E* **52** (1995) 3656.

[9] Fisher D.S., *Phys. Rev. Lett.* **56** (1986) 1964.

[10] Balents L. and Fisher D.S., *Phys. Rev. B* **48** (1993) 5949.

[11] Mézard M. and Parisi G., *J. Phys. I France* **1** (1991) 809; *J. Phys.* **A23** (1990) L1229.

[12] Bouchaud J.P., Mézard M. and Yedidia J., *Phys. Rev. B* **46** (1992) 14686.

[13] Giamarchi T. and Le Doussal P., *Phys. Rev. B* **52** (1995) 1242.

[14] Parisi G., *J. Phys. France* **51** (1990) 1595.

[15] Mézard M., *J. Phys. France* **51** (1990) 1831.

[16] Fisher D.S. and Huse D., *Phys. Rev. B* **43** (1991) 10728.

[17] Hwa T. and Fisher D.S., *Phys. Rev. B* **49** (1994) 3136.

[18] Fisher D.S. and Huse D.A., *Phys. Rev. Lett.* **56** (1986) 1601; *Phys. Rev. B* **38** (1988) 373.

[19] Bouchaud J.P., *J. Phys. I France* **2** (1992) 1705; Bouchaud J.P. and Dean D.S., *J. Phys. I France* **5** (1995) 265; See also: Monthus C. and Bouchaud J.P., preprint cond-mat 9601012, to appear in *J. Phys. A*.

[20] Dotsenko V.S., Feigel'man M.V. and Ioffe L.B., Spin-Glasses and related problems, Soviet Scientific Reviews, vol. 15 (Harwood, 1990).

[21] Nattermann T., Stepanow S., Tang L.-H. and Leschhorn H., *J. Phys. II France* **2** (1992) 1483.
[22] Narayan O. and Fisher D.S., *Phys. Rev. B* **46** (1992) 11520.
[23] Ertas D. and Kardar M., *Phys. Rev. Lett.* **73** (1994) 1703.
[24] $\frac{1}{N}$ corrections have been considered in [25] and in Carlucci D., De Dominicis C. and Temesvari T., in press.
[25] Mézard M. and Parisi G., *J. Phys. I France* **30** (1992) 2231.
[26] For a Gaussian variational theory without replicas, see the works of Orland H., *Europhys. Lett.* **27** (1994) 317; and Garel T., Iori G. and Orland H. *Phys. Rev. B*, to appear.
[27] Mézard M., Parisi G. and Virasoro M.A., Spin Glass Theory and Beyond (World Scientific, Singapore 1987).
[28] Larkin A. and Ovchinnikov Y., *J. Low. Temp. Phys.* **34** (1979) 409.
[29] Bouchaud J.P., Carlucci D., De Dominicis C. and Mézard M., in preparation.
[30] This can actually be understood within the general context of extreme event statistics [31].
[31] Bouchaud J.P. and Mézard M., in preparation.
[32] Burgers J.M., The Non-Linear Diffusion Equation (D. Reidel Pub. Co., 1974).
[33] Kida S., *J. Fluid. Mech.* **93** (1979) 337.
[34] Feigel'man M. and Ioffe L., *Z. Phys. B* **51** (1983) 237.
[35] Removing all the modes $k > k_0$ in the replica calculation leads to a correlation function $R_{\rm RSB}(y)$ indeed dominated by the vicinity of $u(k_0) \propto k_0^{\theta}$.
[36] Fisher D.S., Henley C. and Huse D.A., *Phys. Rev. Lett.* **55** (1985) 2924.
[37] This is because a potential with multiple cusps generically has multiple wells with non-uniformly separated and unequal minima, so that repeated minimization allows the surviving modes to remain only in the deepest well.
[38] For experimental arguments in favour of a hierarchical organisation of phase-space, see: Vincent E., Hammann J. and Ocio M., in "Recent Progress in Random Magnets", D.H. Ryan Ed. (World Scientific Pub. Co. Pte. Ltd, Singapore 1992) p. 207.
[39] Infinite N properties have been thoroughly investigated in [40] but might belong to a different dynamical class since activated effects disappear, at least for short range correlations.
[40] Cugliandolo L. and Kurchan J., *Phys. Rev. Lett.* **71** (1993) 173; Cugliandolo L. and Le Doussal P., *Phys. Rev.* **E 53** (1996) 1525; Cugliandolo L., Le Doussal P. and Kurchan J., *Phys. Rev. Lett.* **76** (1996) 2390; Franz S. and Mézard M., *Europhys. Lett.* **26** (1994) 209; *Physica* **A209** (1994) 1.
[41] Ioffe L.B. and Vinokur V.M., *J. Phys. C* **20** (1987) 6149.
[42] Drossel B. and Kardar M., *Phys. Rev.* **E52** (1995) 4841.
[43] Yoshino H., Off-Equilibrium Dynamics of 1+1 dimensional Directed Polymer in Random Media, preprint cond-mat 9510024.