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Short Communication

Phase Space Diffusion and Low Temperature Aging

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Abstract. — We study the dynamical evolution of a system with a phase space consisting of configurations with random energies. The dynamics we use is of Glauber type. It allows for some dynamical evolution and aging even at very low temperatures, through the search of configurations with lower energies. This simple model provides an example of a new type of mechanism for the aging effect.

Many efforts are being devoted to the study of the out-of-equilibrium dynamics of spin-glasses, in order to understand several experimental findings, such as aging and the joint existence of a new dynamics and memory in temperature cycling experiments [1]. Roughly speaking, the aging effect is observed in the decay of the thermoremanent magnetization (TRM), where this decay is found to depend on the age of the system (the time it has spent in the low temperature phase) in all the accessible time regimes. It implies that the behaviour of the system depends on all its history, and that non-stationary dynamics persists for arbitrary long time-scales. In a first approximation, the decay which is observed experimentally or numerically is often well described by the fact that the TRM measured after a duration time τ after turning off the magnetic field depends on the age t_w, roughly like a function of τ/t_w. Various approaches have been used to try to understand this effect, including some phenomenological studies based on droplet or domain growth [2], numerical simulations [3], mean-field models [4–6], diffusion on tree like structures [7], and models based on the existence of traps in phase space.

In this note we shall elaborate along the lines of this last approach. It deals directly with the structure of the phase space, made of many metastables states, figuring the configurations of a spin system. Such a picture has been put forward [8], and generalized [9], presenting the phase space as a random energy landscape made of traps, with a broad distribution of trapping times: the energies are the low lying ones of a random energy model (REM) [10], so they have an exponential probability distribution, and the trapping times, given by an Arrhenius law, have a power-law distribution with infinite mean. The dimension of the space is infinite (equivalently,

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all the traps are connected). In this frame, the diffusion is anomalous [8,9,11], and aging is present. One virtue of this approach has been to point out a simple "kinematic" ingredient for aging: the distribution of trapping times itself does not evolve in time, but if this distribution is broad (i.e., it is a Lévy law with an infinite mean trapping time), the probability to be at time t_w in a trap of lifetime τ depends on the number of visited traps and thus on t_w, which induces aging. Besides this kinematic effect, there might well exist another, "dynamical", source of aging, namely the explicit evolution with time of the distribution of trapping times. We shall provide hereafter such an example of a dynamical aging process.

In the trap model, the probability of hopping from one configuration i to another configuration j, $W_{i\to j}$, depends only on the energy of the site i: the energies are in fact seen as energy barriers, which are then uncorrelated. The model can be solved in full details: the equilibrium correlation decays in time as a power law [12]; however this equilibrium correlation is never observed (for an infinite system) because of an aging effect due to a broad distribution of trapping times [8]. The equilibrium dynamics of the REM has been studied in other models, corresponding to some factorized choices of the transition probabilities $W_{i\to j}$ [13,14]. These choices allow for a solution of the master equation, and the equilibrium correlation function displays various forms, including power law and stretched exponential relaxation. Hereafter we shall consider a case where the transition probability is of Glauber type: $W_{i\to j}$ is not factorized, and the energy barriers are now correlated. This seems *a priori* a more refined way to define a dynamics for the REM. Furthermore this case allows for the existence of a dynamical evolution even at zero temperature, through the search of configurations with lower energies. During this evolution it becomes more and more difficult for the system to find a lower configuration, which results in a slow down of the dynamics. We shall show through an explicit solution at zero temperature that this mechanism gives rise to an aging effect in which the system never reaches equilibrium. This aging does not take its origin in energetic barriers, but rather in some kind of entropic barrier, namely the low probability of finding a favourable direction in phase space. Another example of aging due to entropic barrier has been proposed recently by Ritort [15].

The model we consider is defined as follows: The system can be in any of N configurations $i = 1,..,N$. The configuration i has an energy E_i. The energies are independent random variables with distribution $P(E)$. The probability of hopping from one configuration to another can be defined in several ways, the only a priori constraint being the detailed balance:

$$W_{i\to j}\mathrm{e}^{-\beta E_i} = W_{j\to i}\mathrm{e}^{-\beta E_j} \tag{1}$$

For example, for the trap model, where the lifetime of configuration i is $\tau_i = \exp(-\beta E_i)$, the transition rates are $W_{i\to j} = \dfrac{\exp(\beta E_i)}{N}$ Here we consider a transition probability depending on both E_i and E_j, given by the Glauber dynamics:

$$W_{i\to j} = \frac{1}{N}\,\frac{1}{1+\exp(\beta(E_j - E_i))} \tag{2}$$

As mentioned before, this system is quite different from the trap model (see Fig. 1). For instance, at zero temperature, the jump to a lower state is allowed in our model, while it is impossible to jump out of a trap.

In this study, we will be interested in computing the law of diffusion (the number of configurations reached at time t), the evolution of the mean energy with time, the probability, given a time t, to be in a configuration of lifetime τ (we shall define the lifetime τ precisely below), which we will note $p_t(\tau)$, and the two-times correlation function $C(t_w + t, t_w)$. This

Fig. 1. — a) trap model; b) "step" model. Remember that the connectivity is infinite, so that such a picture can be misleading!

last quantity is defined as the mean overlap between the positions of the system at times t_w and $t_w + t$: the overlap is simply either 1 if the system is in the same configuration, or 0 if it has moved. Assuming an exponential decay out of the configurations, the correlation function is related to $p_t(\tau)$ through:

$$C(t_w + t, t_w) = \int_0^\infty \mathrm{d}\tau \; p_{t_w}(\tau) \mathrm{e}^{-t/\tau} \tag{3}$$

Before turning to the exact solution of the master equation at zero temperature, it is useful to start with a discussion of the trapping time distributions. When the system is in configuration i, with energy E_i, the probability of going away per unit time is

$$p_s(E_i) = \sum_j W_{i \to j} \; . \tag{4}$$

For large N and using the definition (2) of the transition probability, one gets at zero temperature:

$$p_s(E_i) = \int_{-\infty}^{E_i} \mathrm{d}E' \; P(E') \tag{5}$$

The "trapping time" τ_i is defined as $\frac{1}{p_s(E_i)}$. It depends only on the energy of the configuration, through the relation:

$$\frac{1}{\tau^2} \frac{\mathrm{d}\tau}{\mathrm{d}E} = P(E) \; . \tag{6}$$

We deduce that, *regardless of P(E)*, the *a priori* distribution of the lifetimes is:

$$P_0(\tau) d\tau = \frac{\mathrm{d}\tau}{\tau^2} \theta(\tau - 1) \tag{7}$$

As in the trap model, this is a broad distribution with a divergent mean lifetime (although here it is just marginally divergent, for any $P(E)$). As was shown in reference [8], this fact in itself creates an aging effect. In our case there is an additional effect because the effective distribution of lifetimes evolves with time. After k jumps the system will be in a lower energy configuration, and the probability $P_k(\tau)$ of having a lifetime τ is different from $P_0(\tau)$. The zero temperature dynamics gives the recursion relation:

$$P_{k+1}(\tau) = P_0(\tau) \int \mathrm{d}\tau' \; \tau' P_k(\tau') \theta(\tau - \tau') \tag{8}$$

which leads to:

$$P_k(\tau) = \frac{(\log\tau)^k}{k!\,\tau^2}\theta(\tau-1)\ . \tag{9}$$

The typical lifetime thus increases exponentially with k (it means that the diffusion is logarithmic). This will add up to the usual effect of a diverging mean lifetime in order to induce aging.

We now proceed to the solution of the master equation at zero temperature using the Laplace transform. Denoting by $p_i(t)$ the probability of being on configuration i at time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}p_i(t) = \sum_j T_{ij}\ p_j(t) \tag{10}$$

with $T_{ij} = W_{j\to i}$ for $i \neq j$, and $\sum_i T_{ij} = 0$. The Laplace transform $\tilde{p}_i(\phi) = \int_0^\infty \mathrm{d}t\ p_i(t)\mathrm{e}^{-\phi t}$ satisfies the equation:

$$\tilde{p}_i(\phi) = \frac{p_i(0) + \frac{1}{N}\sum_j\ \theta(E_j - E_i)\tilde{p}_j(\phi)}{\phi + \dfrac{1}{\tau_i}} \tag{11}$$

where the lifetime τ_i is defined as before by $\dfrac{1}{\tau_i} = \sum_{j\neq i} T_{ji}$ and where we will take $p_i(0) = \dfrac{1}{N}$. To solve this equation we introduce the Laplace transform of the occupation probabilities for all configurations of energy E:

$$f(E,\phi) \equiv \sum_j \tilde{p}_j(\phi)\delta(E - E_j)\ . \tag{12}$$

Using equation (11) one derives for f the equation:

$$f(E,\phi) = g(E,\phi)\left(1 + \int_E^\infty \mathrm{d}E' f(E',\phi)\right) \tag{13}$$

where

$$g(E,\phi) = \frac{1}{N}\sum_j \frac{\delta(E-E_j)}{\phi + \frac{1}{\tau_j}} = \frac{P(E)}{\phi + \frac{1}{\tau(E)}} \tag{14}$$

The self-consistency equation (13) is easily solved and gives:

$$f(E,\phi) = g(E,\phi)\exp\left(\int_E^\infty g(E',\phi)\mathrm{d}E'\right) \tag{15}$$

We can now use this solution to compute the physical quantities of interest. We start with the probability $p_t(\tau)$ to be at time t in a configuration of lifetime τ. Its Laplace transform with respect to t is given by:

$$p_\phi(\tau) = \sum_i \tilde{p}_i(\phi)\delta(\tau - \tau_i) = \frac{P_0(\tau)}{\left(\phi + \frac{1}{\tau}\right)}\frac{f(E(\tau),\phi)}{g(E(\tau),\phi)} = \frac{\phi + 1}{(1+\phi\tau)^2} \tag{16}$$

This Laplace transform can be inverted and gives:

$$p_t(\tau) = \frac{t\tau - t + \tau}{\tau^3}\exp\left(-\frac{t}{\tau}\right)\theta(\tau - 1) \tag{17}$$

We see that this expression decreases as t/τ^2 for $t << \tau$, as for a model of traps for $P_0(\tau) = 1/\tau^2$, but the exponential term makes the probability of being at time t in a configuration with lifetime smaller than t very small.

The correlation function can also be computed for large t and t_w:

$$C(t_w + t, t_w) \simeq \frac{t_w}{t_w + t} \tag{18}$$

We see immediately the t/t_w scaling of the correlation function. The behaviour $\lim_{t\to\infty} C(t_w + t, t_w) = 0$, sometimes called "weak ergodicity breaking" [8] shows the existence of a non equilibrium dynamics even at zero temperature, while the behaviour in the other limit, $\lim_{t_w\to\infty} C(t_w + t, t_w) = 1$, reflects the fact that the equilibrium dynamics is frozen at $T = 0$.

Let us emphasize that all these results are independent of the distribution $P(E)$. The two main hypotheses of the derivation are the fact that the connectivity is infinite, and the temperature has been taken equal to zero. These results can be partially extended in the case where $P(E)$ is an exponential distribution, $P(E) \sim \rho \exp(\rho E)\theta(-E)$. Such a distribution has been found in mean field spin glass models [16], and it is at the heart of the trap model description, since it leads to a broad distribution of lifetimes when $\beta > \rho$. Specializing in this case, we can first explain in more detail the zero temperature dynamics studied above. Indeed, the relation between energy and trapping time can be explicited as: $\tau_i = \mathrm{e}^{-\rho E_i}$ and the energy distribution evolves then as

$$P_k(E) = \frac{(-\rho)^{k+1} E^k}{k!} \exp(\rho E)\theta(-E) \tag{19}$$

The mean energy decreases as $-k/\rho$ with the number of visited configurations, or equivalently as $-\log t/\rho$ (remember that the diffusion is logarithmic).

For finite temperature, the set of self-consistent equations is more complicated; we write $a_i = \exp(\beta E_i)$, so that

$$\frac{1}{\tau_i} = a_i^{\rho/\beta} \int_0^{a_i^{-\rho/\beta}} \frac{\mathrm{d}v}{1 + v^{\beta/\rho}} \tag{20}$$

and we obtain

$$\tilde{p}_i(\phi) = \frac{p_i(0) + \displaystyle\int_0^\infty \mathrm{d}\lambda \mathrm{e}^{-\lambda a_i} f(\lambda, \phi)}{\phi + \frac{1}{\tau_i}} \tag{21}$$

$$f(\lambda, \phi) = g(\lambda, \phi) + \int_0^\infty \mathrm{d}\mu \; g(\lambda + \mu, \phi) f(\mu, \phi) \tag{22}$$

$$g(\mu, \phi) = \frac{1}{N} \sum_j \frac{a_j \mathrm{e}^{-\mu a_j}}{\phi + \frac{1}{\tau_j}} = \frac{\rho}{\beta} \int_0^1 \mathrm{d}u \frac{u^{\rho/\beta} \mathrm{e}^{-\mu u}}{\phi + u^{\rho/\beta} \displaystyle\int_0^{u^{-\rho/\beta}} \frac{\mathrm{d}v}{1 + v^{\beta/\rho}}} \tag{23}$$

Taking $\lambda = \tau^{\beta/\rho}$ and writing $g(\tau, \phi)$, $f(\tau, \phi)$ instead of $g(\tau^{\beta/\rho}, \phi)$, $f(\tau^{\beta/\rho}, \phi)$, we obtain for f the following scaling:

$$f(\tau, \phi) = \frac{\rho}{\beta} \tau^{1-\frac{\beta}{\rho}} h(\phi\tau) \tag{24}$$

with $h(x)$ behaving as $1/x^2$ for $x >> 1$ and $\int_0^\infty \mathrm{d}x h(x)$ finite. After some calculations, it can be shown that $p_t(\tau)\mathrm{d}\tau$ behaves like $\frac{\mathrm{d}\tau \; t}{\tau^2}$ for $1 << t << \tau$, and as $\frac{\mathrm{d}\tau}{\tau} \left(\frac{\tau}{t}\right)^{1+\frac{\beta}{\rho}}$ for $1 << \tau << t$, and for $\beta >> 1$.

We obtain thus qualitatively the same behaviour for $p_t(\tau)\mathrm{d}\tau$ as before, and also for the correlation function: the dynamics is not modified by a small temperature.

Note that this behaviour holds in the limit of infinite N; for any finite N the system eventually thermalises, after a time proportional to N (for example, the minimal energy of N states with exponentially distributed energies is $-\log N/\rho$ so it takes a time N to find it).

For this model, some of our results are similar to those of reference [9]: the mean energy decreases as $-\log(t)$, and at time t the most probable configurations are of lifetime $\tau = t$. Nevertheless we must emphasize that the mechanism is totally different: in a model of traps, the mean energy decreases because the system visits more and more traps, and so it has more and more chances to find deep ones. At each step $P_k(E)$ remains the same: $P_k = P_0$. The diffusion is in $t^{\rho/2\beta}$, and the energy at step k evolves like $-\log(k)$, because the minimum of k energies distributed according to the exponential distribution is in $-\log(k)$. In our model, on the opposite, the energy distribution that the particle sees evolves, and so does the distribution of trapping times (see Eq. (9)); the energy decreases in fact because it is easier to find a configuration with lower energy (it takes a time proportional to $\exp(-\rho E)$) than to move by thermal activation (a time $\exp(-\beta E)$ is needed). It means that the system spends less time in a given configuration but, since the energy at step k decreases like $-k$, the diffusion is much slower (logarithmic instead of a power-law), so we finally get the same behaviour for the mean energy as a function of time. However the main feature is that there exists a zero-temperature dynamic, which is qualitatively not modified by a small temperature.

It would of course be very interesting to be able to generalize this approach beyond the case of an infinite connectivity. For instance a more realistic definition of the REM dynamics could be to start from the definition of the REM in terms of spins with p ($\to \infty$) spin interaction [10,17], and use the transition matrix resulting from single spin flip dynamics. This seems rather complicated at the moment.

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