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A variational approach to directed polymers

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Abstract. In this paper we develop a variational approach for directed polymers in random media in the framework of the replica formalism. We derive the functional expression for the free energy in terms of the probability distribution of the probability of finding a polymer at a given point. We compare these results with those coming from the broken replica symmetry approach and we present some new computations in this framework.

1. Introduction

A crucial open problem in statistical mechanics consists in finding the behaviour of fluctuating manifolds in a random medium with quenched disorder (for a review see Nattermann and Rujan 1989); recent progress has been made using the replica formalism (Derrida and Gardner 1986, Kardar 1987), and the problem was apparently solved in the limit of infinitely many dimensions where a simple Flory law has been found to be correct (Mézard and Parisi 1990, 1991, hereafter referred to as MP). The Flory results cannot be valid in all dimensions for all kinds of correlations of the noise (e.g. they are wrong for directed polymers with short range noise in one dimension). Systematic corrections can, in principle, be computed in the replica framework (1-D expansion), but they are technically rather involved and this has not yet been done (for a first step in that direction see Mézard and Parisi 1992). It is therefore useful to explore other approaches. In this note we concentrate our attention on directed polymers which, due to their one-dimensional structure are rather simpler than the fluctuating manifold with generic dimension.

Indeed in each instance of the directed polymers problem we can compute the probability $\rho(x)$ that the polymer ends at the point x. The solution of the statistical problem consists in finding the probability distribution $P[\rho]$, i.e. the probability distribution (for different instances) of the probability ρ.

In section 2 we derive a variational principle for $P[\rho]$ which reduces the problem to finding the maximum of the functional P. In section 3 we derive some consequences of this variational principle. In section 4 we compare this approach with the broken replica symmetry formalism. In section 5 we present a simple computation of $P[\rho]$ in the broken replica approach and finally in section 6 we discuss the zero-dimensional limit (named the *toy model*).

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2. The Schrödinger equation for zero particles

The problem may be formulated as follows (see for instance Kardar and Zhang 1987). In the continuum limit the classical Hamiltonian is

$$H = \int \mathrm{d}l \left\{ \tfrac{1}{2} \sum_{\nu=1,\dots,N} [\partial \omega_\nu(l)]^2 + \eta(l, \omega(l)) \right\} \tag{2.1}$$

where $\omega(l)$ is a vector with N components. The vector valued function $\omega(l)$ represnts the coordinates of a one-dimensional manifold in a $(N+1)$-dimensional space. The variable l parametrizes the manifold, which has coordinates $(l, \omega(l))$. The probability of a configuration $\omega(l)$ is $P[\omega] \propto \exp(-H[\omega])$.

The function $\eta(l, y)$ (y being an N-dimensional vector) represents the effect of the disorder and it is a quenched variable, which is usually supposed to be Gaussian distributed. Different models may be obtained by choosing different forms for the correlation of the noise. In this note we consider only the case where

$$\begin{aligned} &\langle \eta(l_1, y_1) \eta(l_2, y_2) \rangle = \delta(l_1 - l_2) V(y_1 - y_2) \\ &V(x) \equiv g / \lambda N^{(1+\lambda/2)} |x|^{-\lambda} \end{aligned} \tag{2.2}$$

g being the coupling constant. Hereafter we indicate with $\langle A \rangle$ the average of $A[\eta]$ with respect to the noise η.

When $\lambda = N$, the RHS of (2.2) becomes proportional to $\delta(y_1 - y_2)$ and the noise is short range (Medina *et al* 1989, Halpin-Healey 1989). The models we consider are thus characterized by two parameters (N and λ); for some values of these parameters the model may not be defined in the continuum and a cutoff must be introduced (e.g. a cutoff at short distances must be introduced if $\lambda \geqslant 2$).

The long range behaviour of these models at low temperatures is characterized by a single exponent ζ, which describes the growth of the transverse fluctuations of the manifold as function of the distance, i.e.

$$\langle [\omega(l_1) - \omega(l_2)]^2 \rangle \propto |l_1 - l_2|^{2\zeta} \tag{2.3}$$

for large $|l_1 - l_2|$. Using scaling invariance and Galilean invariance the values of the other exponents may be related to ζ (Huse *et al* 1985, Kardar and Zhang 1987).

The advantage of directed polymers (with respect to the general case where l is a vector) is that in this case the problem may be formulated in a slightly different way.

We consider the stochastic differential equation

$$\partial G(l, x)/\partial l = \tfrac{1}{2} \Delta G(l, x) + \eta(l, x) G(l, x). \tag{2.4}$$

We fix the boundary conditions at the origin, i.e. at $l = 0$ (the most popular conditions are: (*a*) $G(0, x) = 1$; or (*b*) $G(0, x) = \delta(x)$); at some it may be convenient to restrict the equation to the interval $-L/2 < x_\nu < L/2$, $\nu = 1, \dots, N$.

Although G depends on η, we have not indicated explicitly this fact in order to lighten the notation. In the following the average over η will be denoted by a bar and the function $\rho(l, x) \equiv G(l, x) / \int \mathrm{d}y\, G(l, y)$ is introduced. The problem consists in computing the probability distribution of the function ρ or equivalently its moments. In other words we must find a functional $P_l[\rho]$ such that

$$\int \mathrm{d}[\rho] P_l[\rho] A[\rho] = \langle A[\rho(l, x)] \rangle \tag{2.5}$$

where A is a generic functional and $d[\rho]$ denotes functional integration. In this paper we will restrict ourselves to the computation of $P_l[\rho]$ in the equilibrium limit where l goes to infinity; from here on $P[\rho]$ will denote $\lim_{l\to\infty} P_l[\rho]$.

In one dimension (i.e. $N=1$), if the noise has a delta function correlation (white noise), the model is soluble and it is known that

$$P[\rho]=\exp\left(-C\int dx\{d(\ln \rho(x))/dx\}^2\right). \tag{2.6}$$

This is a rather fortunate case.

In the generic case it is not easy to find the form of the function $P[\rho]$. We will show now that $P[\rho]$ does satisfy a variational principle which can be used to find an approximated form the functional $P[\rho]$.

The first step consists in using the replica formalism and in integrating over the Gaussian noise η. We introduce n copies of the N-dimensional vector ω, which now carries two indices: ω_a^α; $\alpha=1,\ldots,N$; $a=1,\ldots,n$. The expected value of the partition function to the nth power can be obtained by using the following classical Hamiltonian:

$$H_R=\int dl\left\{\sum_{a=1,n}\sum_{\alpha=1,N}\tfrac{1}{2}[\partial\omega_a^\alpha(x)]^2-\sum_{a,b=1,n}V(\omega_a(l)-\omega_b(l))\right\}. \tag{2.7}$$

Apart from the simple case where $\lambda=-2$ (Parisi 1990a), the model is not explicitly soluble. It is interesting to note that the model is invariant under the group $O(N)$ (i.e. rotations in physical space), under the S_n group (i.e. the permutation group of n replicas) and under the group of translations. The actual symmetry group is larger (it contains rotations which mix the l and the ω coordinates), but we will not use it. As usual, we have eventually to consider the limit n going to zero, which gives a distinctive flavour to the replica approach (Mézard *et al* 1987).

The classical Hamiltonian (2.7), can be also considered as the Euclidean (i.e. imaginary time) action in the Feynman-Katz representation for n interacting particles in the presence of the attractive potential $-V$. The ground-state wavefunction $\psi(x)$ satisfies a Schrödinger equation corresponding to the quantum Hamiltonian

$$\boldsymbol{H}=-\tfrac{1}{2}\sum_{a=1,n}\Delta_a-\sum_{a=1,n,b=1,n}V(x_a-x_b), \tag{2.8}$$

where Δ_a stands for the Laplacian with respect to the ath variable.

The ground-state function is rather important because it gives the probability distribution of the final point of the trajectories for large l. Indeed in the infinite l limit we have

$$\overline{\rho(x_1)\rho(x_2)}\equiv\int dP[\rho]\rho(x_1)\rho(x_2)\propto\int dx_3\ldots dx_n\,\psi(x_1,\ldots,x_n). \tag{2.9}$$

Similar relations hold for higher moments of ρ.

While it is relatively simple to compute the wavefunction for n particles (provided that n is not too large) the situation is not easy when $n=0$: a direct numerical approach is not possible. Our strategy here consists in representing the function ψ as

$$\psi(x_1,\ldots,x_n)\propto\int dR[\mu]\mu(x_1),\ldots,\mu(x_n). \tag{2.10}$$

Using functional techniques it can be proved that for attractive potentials (i.e. V positive) the ground-state wavefunction ψ satisfies (2.10), where the measure $dR[\mu]$

is concentrated on non-negative functions $\mu(x)$. The relationship between R and P is simple:

$$\overline{\rho(x_1)\rho(x_2)} \equiv \int \mathrm{d}P[\rho]\rho(x_1)\rho(x_2) = \int \mathrm{d}R[\mu]\mu(x_1)\mu(x_2)/I[\mu]^2 \tag{2.11}$$

where $I[\mu] \equiv \int \mathrm{d}x\, \mu(x)$.

We want to use the quantum mechanics variational principle for ψ (i.e. ψ minimizes $\langle\psi|\boldsymbol{H}|\psi\rangle$ under the condition $\langle\psi|\psi\rangle = 1$). We find that

$$\langle\psi|\boldsymbol{H}|\psi\rangle = \int \mathrm{d}R[\mu]\,\mathrm{d}R[\sigma]\mathfrak{H}_n[\mu,\sigma] \tag{2.12}$$

where

$$\mathfrak{H}_n[\mu,\sigma] = n/2 \int \mathrm{d}x \sum_{\nu=1,N} \mathrm{d}\mu/\mathrm{d}x_\nu\, \mathrm{d}\sigma/\mathrm{d}x_\nu\, I[\mu\sigma]^{n-1}$$
$$- n(n-1) \int \mathrm{d}x\, \mathrm{d}t\, \mu(x)\mu(y)\sigma(x)\sigma(y)V(x-y)I[\mu\sigma]^{n-2} \tag{2.13}$$

and

$$\langle\psi|\psi\rangle = \int \mathrm{d}R[\mu]\,\mathrm{d}R[\sigma]I[\mu\sigma]^n. \tag{2.14}$$

The Schrödinger equation (which can be obtained by the variation principle) becomes in this new representation:

$$\int \mathrm{d}R[\sigma]\mathfrak{H}_n[\mu,\sigma] = E_n \int \mathrm{d}R[\sigma]I[\mu\sigma]^n \tag{2.15}$$

where (2.15) should be valid for any (sufficiently regular) function $\mu(x)$.

It is easy to check that for integer n the Schrödinger equation is a polynomial of degree n in the functions μ. If we functionally differentiate it with respect to $\mu(x_1)\ldots\mu(x_n)$, we find the usual Schrödinger equation, which involves only the moments of order n of R. This simplification is absent for non-integer n.

We can now easily continue our equations in n up to $n = 0$. In this limit the variational principle can be expressed as finding the stationary point of $F[R]$ defined by

$$F[R] \equiv \int \mathrm{d}R[\mu]\,\mathrm{d}R[\sigma]\mathfrak{H}[\mu,\sigma] \tag{2.16}$$

where

$$\mathfrak{H}[\mu,\sigma] = \tfrac{1}{2}\int \mathrm{d}x \sum_{\nu=1,N} (\partial_\nu\mu)(\partial_\nu\sigma)I[\mu\sigma]^{-1}$$
$$+ \int \mathrm{d}x\, \mathrm{d}y\, \mu(x)\mu(y)\sigma(x)\sigma(y)V(x-y)I[\mu\sigma]^{-2} \tag{2.17}$$

under the condition $\int \mathrm{d}R[\mu] = 1$. The Schrödinger equation ($\frac{1}{2}\delta F/\delta R[\mu] = E$) now becomes

$$\int \mathrm{d}R[\sigma]\mathfrak{H}[\mu,\sigma] = E \tag{2.18}$$

where

$$E = \lim_{n\to 0} E_n/n. \tag{2.19}$$

$\mathfrak{H}[\mu, \sigma]$ is homogeneous of degree zero in σ and μ. Therefore we can consider in (2.18) that μ is normalized ($\int \mu(x)\,dx = 1$), and that the integration measure on σ is restricted to normalizable $\sigma(x)$ functions. As μ and σ are positive, they can be considered as probability distributions. In such a subspace the measure $dR[\mu]$ and $dP[\rho]$ coincide (see (2.11)), and (2.18) can be interpreted as an equation for $dP[\rho]$:

$$\int dP[\rho]\mathfrak{H}[\mu, \rho] = E \tag{2.20}$$

where ρ and μ are probability distributions.

Equation (2.20) is an exact statement, which is the main result of this paper. In principle it could also have been proved by a directed probabilistic method, but we have not tried to follow that approach.

3. The consequences of the Schrödinger equation

We have not explored all the implications of the functional representation of the Shrödinger equation. We will, here, make only a few observations on some consequences of the Schrödinger equation.

In principle (2.18) must be satisfied for any μ. It may be convenient to take $\mu(x) = \mu_0(x) + \varepsilon(x)$. We can thus consider the function derivatives with respect to ε at $\varepsilon = 0$ for $\mu_0(x) = 1$. In this way we obtain some relations among the ρ correlation functions. For example the first two equations are

$$\int dy\, V(z_1 - y)G_2(z_1, y) - \int dy\, dx\, V(x-y)G_3(x, y, z_1) = 0 \tag{3.1}$$

and

$$-\Delta G_2(z_1, z_2)$$

$$= 2V(z_1 - z_2)G_2(z_1, z_2) - 4\int dy[V(z_1 - y) + V(z_1 - y)]G_3(y, z_1, z_2)$$

$$+ 6\int dy\, dx\, V(x-y)G_4(x, y, z_1, z_2) \tag{3.2}$$

where the G_n are proportional to the expected n ρs, e.g.

$$G_3(x, y, x) = L^N \int dP[\rho]\rho(x)\rho(y)\rho(z) \equiv \langle \rho(x)\rho(y)\rho(z)\rangle. \tag{3.3}$$

The proportionality factor L^N (L^N being the volume of the space) has been inserted in such a way that we may hope that the Gs have a finite non-zero limit when $L \to \infty$. With this normalization we find $G_1(x) = 1$.

In a similar way one generates a sequence of equations connecting the n-point function with the $(n+1)$ and $(n+2)$-point functions. These equations are very similar in some respects to the Dyson-Schwinger equations of field theory. In the presence of a good anzatz for the high-order Green functions it may be possible to find a self-consistent solution to them.

We do not start this line of investigation here. We only present a few technical remarks.

(1) Equation (3.1) is automatically satisfied. Indeed using translational invariance it turns out to be z_1-independent. After integration on z_1 it can be written as

$$\int dy\, dx\, V(x-y)\langle\rho(x)\rho(y)\rangle = \int dz_1\, dy\, dx\, V(x-y)\langle\rho(x)\rho(y)\rho(z_1)\rangle. \tag{3.4}$$

In the same way the second equation trivializes after integration over z_1.

(2) The correlation functions probably do not satisfy the usual cluster decomposition at large distances. Indeed it is reasonable to suppose that in most cases the function ρ will be nearly concentrated in a compact region, and only in infrequent cases will it have non-negligible contributions in two widely separated regions (Parisi 1990b, Mézard 1990). The correlation functions at large distance will therefore be dominated by rare events.

Let us be quantitative by trying to compute $\langle\rho(x+z_1)\dots\rho(x+z_k)\rho(y+z_1')\dots\rho(y+z_{k'}')\rangle$ in the region of large $x-y$ with the zs kept finite (for simplicity let us absorb a factor V in the definition of the average $\langle\cdot\rangle$). The extension of the analysis of Parisi (1990b) suggests that the relative shapes of the function ρ around the points x and y are uncorrelated for large $x-y$, but there is a probability $C(x-y)$ that both ρs are of order one. In this case one finds that when $x-y$ is large

$$\begin{aligned}\langle\rho(x+z_1)\dots\rho(x+z_k)\rho(y+z_1')\dots\rho(y+z_{k'}')\rangle \\ \approx\langle\rho(x+z_1)\dots\rho(x+z_k)\rangle\langle\rho(y+z_1')\dots\rho(y+z_{k'}')\rangle \\ \times A_{k,k'}C(x-y)\end{aligned} \tag{3.5}$$

where a more detailed analysis is needed to find the values of the constants $A_{k,k'}$; $C(x-y)$ is proportional to $G_2(x-y)$ and according to conventional wisdom it decreases asymptotically with a negative power of $|x-y|$, i.e. as $|x-y|^{-\eta}$, with $\eta = N+\alpha$, α being a critical exponent of the model. As a consequence of the scaling relations at large distances (if any) of the correlation functions of ρ should be of the form:

$$\langle\rho(\Lambda z_1)\dots\rho(\Lambda z_k)\rangle \approx \Lambda^{-(k-1)\eta}\langle\rho(z_1)\dots\rho(z_k)\rangle \tag{3.6}$$

in the region where Λ goes to infinity. Here the exponent at the RHS of (3.6) is given by $(k-1)\eta$, not by the usual $k\eta$. In some sense this change in the exponent is a side effect of the necessity of absorbing the factor V in the definition of the average $\langle\cdot\rangle$.

(3) It is interesting to find out if the different terms in (3.2) scales in the same way. The term at the LHS scales as $1/x$ to the power $2+\eta$. The three terms at the RHS formally scale as $1/x$ to the powers $\eta+\lambda$, $\eta+\lambda+\alpha$, $\eta+\lambda+2\alpha$. In the case where the potential is sufficiently long range and the Flory result is correct, we have

$$3\alpha+\lambda = 2 \tag{3.7}$$

so that cancellations must be present in the RHS of (3.2) (the LHS scales as $\eta+\lambda+3\alpha$). The cancellation of the term with exponent $\eta+\lambda$ may be understood to be a consequence of identities like (3.4), but the cancellation of the remaining terms is more mysterious. It is rather amusing to note that the four terms differ one from the other by a single power of x^α.

(4) Instead of studying the Taylor expansion around $\mu_0(x)=1$ of our functional equation (2.20), different starting points may be considered. In particular the choice

$\mu_0(x)=\delta(x)$ seems particularly interesting. In this case one obtains a sequence of equations, the first two being

$$\tfrac{1}{2}\langle -\Delta\rho(0)/\rho(0)\rangle + V(0) = E \tag{3.8}$$

$$\tfrac{1}{2}\langle -\Delta\rho(x)/\rho(0)\rangle + \tfrac{1}{2}\langle \Delta\rho(0)\rho(x)/\rho^2(0)\rangle + 2(V(x)-V(0))\langle \rho(c)/\rho(0)\rangle = 0. \tag{3.9}$$

These equations can be derived directly from (2.4) by imposing the condition that some quantities are time independent and by using the Ito differential calculus. The first equation states that

$$\partial/\partial l \langle \ln(G(l,x)\rangle = E. \tag{3.10}$$

The second one follows by imposing the condition

$$\partial/\partial l \langle G(l,x)G(l,0)\rangle = 0. \tag{3.11}$$

(5) If it is difficult to find directly a solution using the variational principle, an alternative solution consists in looking for a solution in restricted subspace. A natural possibility is to consider wavefunctions of the form

$$\psi(x_1,\ldots,x_n) = \prod_{1\le a<b\le n} f(x_a - x_b). \tag{3.12}$$

It is easy to see that this factorized wavefunction corresponds to a Gaussian probability distribution for $\varphi(x)\equiv\ln(\rho(x))$:

$$P_g[\varphi] \propto \exp\left(-\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\varphi(x)\varphi(y)g(x-y)\right) \tag{3.13}$$

where the function g is fixed by the condition

$$\int \mathrm{d}z\,\ln(f(x-z))g(z-y) = \delta(x-y) \tag{3.14}$$

or, in other words $\langle\varphi(x)\varphi(y)\rangle = \ln(f(x-z))$.

This surprising relation between the two apparently different quantities can be obtained by writing

$$\psi(x_1,\ldots,x_n) \propto \int \mathrm{d}[\varphi] \exp\left[\sum_{a=1,n}\varphi(x_a) - \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\varphi(x)\varphi(y)g(x-y)\right] \tag{3.15}$$

and performing the explicit Gaussian integral.

The problem now becomes one of finding the function g which makes that functional stationary

$$\int \mathrm{d}P_g[\varphi_1]\,\mathrm{d}P_g[\varphi_2]\tilde{\mathcal{Q}}[\exp(\varphi_1),\exp(\varphi_2)]. \tag{3.16}$$

The functional integral is non-Gaussian and cannot be cone analytically. The only possibility consists in generating the function φ according to the probability distribution $P_g[\varphi]$ and evaluating the expected $\tilde{\mathcal{Q}}$ numerically. This last approach may seem not very elegant; however it gives the correct results for $N=1$ for short range correlations, so that it cannot be excluded *a priori* that it gives the correct answer for the critical exponents.

(6) A question which we have not discussed is if the functional $F[R]$ should be minimized or maximized. This is a crucial question for the replica approach. In usual cases the question is not ambiguous: the requirement is that the Hessian of the free

energy must have no negative eigenvalues in order to apply the saddle point method in a consistent way. Here it is not clear what plays the role of the Hessian; also we are not using a saddle point. There is no doubt that in the replica approach to this problem (described in the next section), where a functional saddle point method is used, the functional F must be maximized and not minimized (as one could naively think). We could also argue that when n particles are present in a translational invariant system, there is one degree of freedom which corresponds to the motion of the centre of mass, and $n-1$ remaining degrees of freedom; in the limit n going to zero, there remain a negative number of effective degrees of freedom (on which the functional depends) and therefore one should maximize, not minimize F (see also Parisi 1990b).

Moreover $E_1 = 0$ and therefore convexity arguments imply that E, defined in (2.19), is non-negative; $E = 0$ can be trivially obtained by the simple wavefunction $\psi = 1$. Therefore a non-zero result for E may be obtained only if we maximize, not minimize F. The consistency of this procedure can be explicitly verified for directed polymer on a lattice in the infinite-dimensional limit (Derrida and Spohn 1988, Cook and Derrida 1989a, b).

4. Breaking the replica symmetry

If we want to pursue the approach of the previous section, a possibility for doing analytic computations consists in using the broken replica symmetry formalism. Indeed the formalism of broken replica symmetry provides a large variety of functionals $P[\rho]$ which have the following properties:

(a) They depend on many parameters so that we have a large variety of choices.

(b) If desired, we can construct these probabilities in such a way that the system is scaling invariant at large distances.

(c) The expectation values of $\tilde{\mathcal{O}}$ and of the ρ correlations can be computed explicitly so that the minimum of the functional F can be found for this form of $P[\rho]$; one can also take the broken replica symmetry solution as the starting point of a perturbative expansion.

(d) Last, but not least, the replica symmetry broken form for $P[\rho]$ seems to be exact when the dimension N of the space goes to infinity.

Let us see how the broken replica symmetry formalism works. The reader must have realized that we can use two quite different languages: we can speak about the x dependence of the wavefunction ψ or we can discuss the form of the probability functional $P[\rho]$. The authors of this paper believe that any statement in one of the two languages may be translated (with sufficient effort) to the other language. However, what can be simply said in one language may sound horribly complicated in the other one.

Let us present the broken replica approach in the language of the wavefunction. As this is basically a rephrasing—in a new language—of the method used in Mézard and Parisi (1991), we shall not give full details. We shall also keep to the case to the case of short-range correlations of the potential, where $V(x) = \delta(x)$. The approach is variational: we start with a class of wavefunctions $\psi_Q(x)$, which depend on some parameters which we denote Q, such that

$$F(Q) \equiv \langle \psi_Q(x) | \boldsymbol{H} | \psi_Q(x) \rangle \tag{4.1}$$

can be computed as far as possible analytically.

The simplest case consists in a Gaussian form for ψ (Shakhnovich and Gutin 1989). We can therefore assume as trial functions

$$\psi_Q(x) \propto \exp\left(-\tfrac{1}{4} \sum_{a=1,n,b=1,n} Q^{-1}_{a,b} x^a_a x^a_b\right). \tag{4.2}$$

The computation of $F(Q)$ can be readily done in the case of short-range interaction (white noise); after some algebra one finds

$$F(Q) = \frac{N}{4}[\mathrm{tr}(Q^{-1})] - \sum_{a=1,n,b=1,n} [2\pi(Q_{a,a} + Q_{b,b} - 2Q_{a,b})]^{-N/2}. \tag{4.3}$$

The problem is to find the matrix Q which maximizes the $F(Q)$; although the original Hamiltonian is symmetric in the interchange of x_a with x_b (i.e. under the action of S_n, the permutation group of n elements which is called here the replica group), one finds that the best matrix is not symmetric under the action of the replica group. Therefore if we limit our search to Gaussian wavefunctions we are forced to break the replica symmetry group. Indeed a matrix $Q_{a,b}$, which has the hierarchical structure common in spin glasses, is believed to be the best solution of the variational problem. Such a matrix may be parametrized in terms of a function $q(u)$ where u belongs to the interval 0–1, and of the diagonal element $\tilde{q}$.

The appearance of the spurious spontaneous breaking of a symmetry in the variational approach is a well known phenomenon; let us recall a simple example. If we try to compute the ground state of the Hamiltonian

$$p^2 - \lambda x^2 + x^4 \tag{4.4}$$

(which is invariant under the transformation $x \to -x$) and we use a variational wavefunction of the form $\psi(x) \propto \exp(A(x-B)^2)$, for sufficient large values of λ, the best result will be given by the asymmetric wavefunction with $B = \pm B^* \neq 0$. The two possible solutions correspond to two different ground states. The variational approach is rather good in this case in finding the energy of the ground state; the correct wavefunction is quite similar to the symmetric combination of the two asymmetric wavefunctions. The two ground states are split by the tunnel effects that are exponentially small in λ.

On the other hand this strange form of matrix Q corresponds to a rather definite probability distribution for the functional $P[\rho]$ which has been discussed in details in MP. If we try to compare the results of the replica approach with the one of the present paper, we should use a replica symmetric wavefunction. More precisely, using the rules of MP, measure $\mathrm{d}P_Q[\rho]$ should be such that

$$\int \mathrm{d}P[\mu]\,\mathrm{d}P[\sigma]\tilde{\mathcal{Q}}[\mu,\sigma] = \langle \psi^S_Q(x)|H|\psi^S_Q(x)\rangle \tag{4.5}$$

where $\psi^S_Q(x)$ is the symmetrized form of $\psi_Q(x)$:

$$\psi^S_Q(x) \equiv \sum_\pi \psi_Q(\pi(x)) \tag{4.6}$$

where the sum runs over all the elements π belonging to S_n.

Indeed the previous equation for the functional $F(Q)$ can, in principle, be computed by expressing $P(\rho)$ as function of Q and evaluating the expected value of $\tilde{\mathcal{Q}}$. This is not exactly what is done in the replica approach, where the original unsymmetrized wavefunction $(\psi_Q(x))$ is used. The computation of the RHS of (4.5) is much more difficult that the RHS of (4.1). A preliminary evaluation suggests that the difference

between (4.1) and (4.5) vanishes when N goes to infinity. Indeed the difference between (4.1) and (4.5) is given by interference terms (i.e. $\langle\psi_Q(x)|H|\psi_Q(\pi(x))\rangle$ with π different from the identity), which are likely to be small in the limit N going to infinity.

For directed polymers (the situation may be different for higher-dimension random manifolds) the question of whether the replica symmetry is browkn spontaneously or not has a clear answer: the replica symmetry is not broken (in a strong sense) because the ground state of a Hamiltonian with a finite number of degrees of freedom must have the same symmetries as the Hamiltonian (a detailed discussion of the distinction between strong and weak replica symmetry can be found in Parisi and Virasoro 1989, Parisi 1990b). Irrespective of this problem, the form for $P[\rho]$ derived using the broken replica symmetry theory is very useful in finding a good variational ansatz.

The breaking of replica symmetry in a hierarchical fashion leads naturally to non-trivial critical exponents. The behaviour of the correlation function at large distance is related to the behaviour of the function $q(u)$ at small u.

The very nature of the variational approximation we have used strongly suggests that the results obtained with the variational approach cannot be exact for all values of N, while they are likely to be correct in the limit of infinite N, at least for γ in an appropriate range (see for instance Natterman 1987 and Amar *et al* 1991). The construction of $1/N$ expansion seems to be feasible, although it may be quite involved. It is quite possible that the ε expansion is simpler. In any case the explicit computation of the first corrections to the mean field approxiation should be very instructive.

Another interesting application of the broken replica symmetry method is the derivation of explicit forms for the correlation functions. Using the equations of MP, one finds

$$\langle\rho(x)\rho(0)\rangle = (2\pi)^{-N/2}\int_0^1 \mathrm{d}u(2s(u))^{-N/2}\exp(-x^2/4s(u)). \tag{4.7}$$

where

$$s(u)\equiv\tilde{q}-q(u). \tag{4.8}$$

Using the fact that the function $s(u)$ behaves as a power law $u^{-2/\alpha}$, one finds the predicted result at large x:

$$\langle\rho(x)\rho(0)\rangle\propto x^{-(N+\alpha)} \tag{4.9}$$

and the exponent α is equal to χ/ζ, in the notation of MP.

A similar, but longer computation, gives

$$\begin{aligned}\langle\rho(x_1)\rho(x_2)\rho(x_3)\rangle &= \tfrac{1}{2}(2\pi)^{-N}\int_0^1 \mathrm{d}u\, u(3s(u))^{-N/2}\exp(((-(x_{12}^2+x_{23}^2+x_{31}^2)/6s(u))\\ &+\tfrac{1}{2}(2\pi)^{-N}\left\{\int_0^1 \mathrm{d}u\int_u^1 \mathrm{d}v(A(u,v))^{-N/2}\right.\\ &\left.\times\exp[-(s(u)x_{12}^2+s(v)x_{13}x_{23})/A(u,v)]+2\text{ perm}\right\}\end{aligned} \tag{4.10}$$

where the term '2 perm.' refers to the two terms obtained from circular permutations of x_1, x_2 and x_3, x_{ik} is defined to be equal to x_i-x_k and $A(u,v)$ is given by $s(v)(4s(u)-s(v))$.

The factorization property (3.5) and the scaling laws (3.6) are valid in this case. We also notice that at large distances the three-point function is not conformal invariant. This is not surprising, if we consider the anomalous form of the scaling laws (3.6).

5. Non-Gaussian wavefunctions

If we want to improve the replica symmetry breaking approach maintaining computability we can try to take more complex wavefunctions, in particular non-Gaussian types. To allow the use of a non-Gaussian wavefunction will only improve the results in the variational approach.

Here we show only how the functional F may be computed; we do not discuss the actual solution of the equations we find. Let us first consider the case where replica symmetry is not broken. A simple translational invariation wavefunction is

$$\psi(x_1, \ldots, x_n) = \int \mathrm{d}y\, g(y-x_1) \ldots g(y-x_n) \tag{5.1}$$

where g is an even function which characterizes the wavefunction.

The computation of $\langle\psi|\boldsymbol{H}|\psi\rangle$ is straightforward for any n. We give it only in the limit n going to zero:

$$F[g] = \lim_{n\to 0} \langle\psi|\boldsymbol{H}|\psi\rangle/n \propto -\tfrac{1}{2}\int \mathrm{d}h \sum_{\nu=1,N} [\partial_\nu \ln(G(h))]^2$$
$$+ \int \mathrm{d}h \left(\int \mathrm{d}x\, \mathrm{d}y\, g(x)g(y)V(x-y)g(h-x)g(h-y)/G(h)^2 \right) \tag{5.2}$$

where

$$G(h) = \int \mathrm{d}x\, g(h-x)G(x). \tag{5.3}$$

The expression for $F[g]$ is rather interesting, because it shows a negative kinetic term (the first one) and a positive attractive potential (the second term). Both the kinetic term and the potential term change sign from the usual ones at $n=1$ (assuming translational invariance): indeed they are zero just at $n=1$. If, in the upside down world of $n=0$ particles, we change sign to F, we find the usual signs. It should be clear now why we would maximize and not minimize F, if reasonable results are to be expected.

The computation becomes more involved if we assume that the replica symmetry is broken. Here we limit ourselves to the case where replica symmetry is broken at the one-step level (Mézard *et al* 1987). This amounts to saying that the n particles are divded into n/m blocks of m particles each. To each index a we can associated a pair of indices (i, α), with i going from 1 to m, α going from 1 to n/m, such that the original index a is given by $i+m(\alpha-1)$. The wavefunction we propose is

$$\psi = \int \prod_{\alpha=1,n/m} \mathrm{d}y_\alpha \left\{ \prod_{i=1,m} g(y_\alpha - x_{i,\alpha}) \right\}. \tag{5.4}$$

This wavefunction corresponds to having blocks of m particles forming bound states.

The computation of the energy functional requires some care; however, after a not too difficult computation one finds, always in the limit n going to zero

$$-F[g,m)=(1-m)\left\{\tfrac{1}{2}\int \mathrm{d}h \sum_{\nu=1,N} G(h)^{m-2}[\partial_\nu G(h)]^2 - \int \mathrm{d}h\left\{\int \mathrm{d}x\,\mathrm{d}y\, g(x)g(y)V(x-y)g(h-x) \times g(h-y)G(h)^{m-2}\right\}\right\} \Big/ \int \mathrm{d}h\, G(h)^m. \tag{5.5}$$

In the limit m going to zero, we recover the previous formulae for the unbroken case.

It would be rather interesting to find the minimum of $-F[g,m)$; in particular, the comparison of the general case with the simpler case $g(x)=\exp(-Ax^2)$ should give some indications on how strongly the results of the replica approach are affected by the Gaussian approximations. This point will not be discussed here; we only present some remarks on the probabilistic interpretation, i.e. on the functional $P[\rho]$ corresponding to these two functions.

Instead of giving the explicit form of $P[\rho]$, it is more convenient to describe a process that generates ρ with the probability $P[\rho]$. For simplicity, we will consider the case where we are confined in a box of size L and consider the limit $L\to\infty$ at the end.

In the first case the solution is rather simple. We have to choose a point y, uniformly distributed, and the function $\rho(x)$ is just given by $g(x-y)$. In other words the approximation consists in assuming a fixed shape for the function ρ, which, however, may be located in an arbitrary position depending on the sample.

The corresponding probability distribution for the second case is less trivial and it can be derived by using the techniques of MP. The procedure is as follows: we generate an infinite set of variables f_α such that the probability of finding a given f_α in the interval $z-(z+\mathrm{d}z)$ is given by $\exp(mz)\,\mathrm{d}z$. We define the quantities

$$w_\alpha=\exp(-f_\alpha)\Big/\left(\sum_\gamma \exp(-f_\gamma)\right)$$

and we associate to each α a vector y_α, taken randomly in the box. The corresponding ρ is given by

$$\rho(x)=\sum_\alpha w_\alpha g(x-y_\alpha).$$

This formula generalizes the one for the more standard one-step replica symmetry breaking where the function g is Gaussian.

Mathematical consistency implies that the result for the free energy can be obtained starting from the probability distribution described in the previous equations and by computing the average of $\tilde{\mathcal{P}}$. The strategy of using the wavefunction representation for computing the free energy seems to give the final results for free energy in the most efficient way (at least in the case where one can neglect the necessity of symmetrizing the ground-state wavefunction).

6. The toy model

A simple but interesting limit of directed polymers arises if we neglect the l dependence

of $\omega(l)$. In this way we simply have

$$H = \eta(\omega) \tag{6.1}$$

where ω is a vector with N components; here the manifold reduces to a point.

The function $\eta(y)$ (y being an N-dimensional vector) is a quenched variable, which is Gaussian distributed. Different models may be obtained by choosing different forms for the correlation V of the noise:

$$\overline{\eta(y_1)\eta(y_2)} = V(y_1 - y_2). \tag{6.2}$$

For a given $\eta(x)$, the probability distribution $\rho(x)$ is given by

$$\rho(x) = \exp(-\eta(x)) \Big/ \int \mathrm{d}y \exp(-\eta(u)). \tag{6.3}$$

In the replica approach we find that this model corresponds to the classical Hamiltonian

$$H_n = - \sum_{a=1,n,b=1,n} V(\omega_z - \omega_b). \tag{6.4}$$

Therefore this gives a probability distribution $\exp(-\Sigma_{a,b} V(\omega_a - \omega_b))$, similar to that discussed at the end of section 3 (see (3.8)). This model is interesting because, in spite of its apparent simplicity, it cannot be solved analytically, even for simple forms of the function $V(x)$. On the other hand the broken replica symmetry approach may be successfully used to derive qualitative and quantitative results for this model. It seems also that this toy model has many structural properties in common with real directed polymers. Some aspects were studied in Schulz *et al* (1988), Mézard and Parisi (1992).

Our aim is to find exact relations among the correlation functions of this model, similar to those found for the directed polymers. The simplest way to derive them consists in using integration by parts: one easily gets the Dyson-Schwinger equations

$$\begin{aligned} &\langle \partial/\partial x_\nu^a A(x) f_\nu^a(x) \rangle_n = 0 \\ &f_\nu^a(x) = \sum_{b=1,n} V_\nu(x^a - x^b) \\ &V_\nu(x) = \partial/\partial x_\nu V(x) \end{aligned} \tag{6.5}$$

where A is an arbitrary function of the xs, and $\langle\ \rangle_n$ is the Boltzmann average with the Hamiltonian (6.3).

It may be convenient to introduce the probability $P[\rho]$ such that

$$\begin{aligned} \langle A \rangle &= \int \mathrm{d}\rho\, P[\rho] \rho(x_1) \ldots \rho(x_n) A(x_1, \ldots, x_n) \\ &= \langle \rho(x_1) \ldots \rho(x_n) \rangle_n A(x_1, \ldots, x_n) \end{aligned} \tag{6.6}$$

If we specialize the previous equation (6.6) to the case in which A is an arbitrary function of x^a and x^c, we find the following equation

$$\partial/\partial x_\nu \langle \rho(x)\rho(y) \rangle - 2 \int \mathrm{d}z\, v_\nu(x-z) \langle \rho(x)\rho(y)\rho(z) \rangle + V_\nu(x-y) \langle \rho(x)\rho(y) \rangle \tag{6.7}$$

which looks like those of the previous section apart from the presence of a first derivative, not a Laplacian.

In principle we could obtain similar equations by differentiating (6.7) and by using again the equation of motion.

A fast way to obtain the result consists in recalling that the function $\psi(x) \equiv \exp(H_n(x))$ satisfies the Schrödinger equation

$$-\sum_{a=1,n} \Delta_a \psi(x) + W(x)\psi(x) = 0 \tag{6.8}$$

where the potential w is given by

$$W(x) = \sum_{a=1,n} \left\{ (f_\nu^a(x))^2 + \sum_{b=1,n} \Delta_a V(x^a - x^b) \right\}. \tag{6.9}$$

In this way we find formulae quite similar to those of section 2, both for the variational principle and the identity among the correlation functions. The main difference is that a three-body interaction is present in this case.

In the case $N=1$, $V(x)=|x|$ the three-body potential disappear and we remain with the simple case where the potential contains only a two-body interaction proportional to a delta-function. The probability distributions for the zero-dimensional case with $V(x)=|x|$ are the same as that of directed polymer with short-range correlations, which is implicit in (2.6).

The toy model seems, therefore, a nice testing ground for developing approximate solutions of the variational principle and of the coupled equations of the form (6.7).

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