Variational theory for the pinning of vortex lattices by impurities

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We derive a variational replica-symmetry-breaking theory for the effect of random impurities on equilibrium properties of general ordered (crystalline) structures, in particular two- and three-dimensional vortex lattices and magnetic bubble films. We investigate the role of a finite correlation length for the random potential and show that the short scale behavior of the correlation functions corresponds to the results of Larkin and Ovchinnikov. For larger scales, we find that the translational correlation functions decay as stretched exponentials with exponents, which take different values: one must distinguish between a regime where the typical displacement is much smaller than the lattice spacing and a regime where it is much larger. We predict that, in the absence of dislocations, long-range orientational order is maintained in three and two dimensions. Our results appear to be in agreement with existing experimental data.

I. INTRODUCTION

In the past few years, considerable progress has been made in describing the influence of quenched impurities on the properties of ordered systems. Theoretical concepts which emerged from the study of spin glasses, random-field magnets, and polymers in random landscapes can now be usefully transposed to many other situations. One example, which we shall study in this paper, is the classic problem of the effect of impurities on the Abrikosov lattice of vortex lines in type-II superconductors. Although many important ideas have been put forward since the seminal paper by Larkin[1] in 1970, the present theoretical description is still not satisfactory, in particular because the widely used and exactly soluble model introduced by Larkin has unphysical features, which we shall describe below. Furthermore, detailed experimental observations are now available through high-quality Bitter decorations[2] of (bulk) high-temperature (HT) superconductors, the interpretation of which might require rather precise theoretical predictions. Indeed, a convincing statistical interpretation of the observed distorted lattice could provide indications on the pinning mechanism (i.e., the nature of the "impurities") and thus a better understanding of the transport properties of these materials. The two-dimensional (2D) (thin film) version of this system is closely related to lattices of magnetic bubbles on disordered substrates, for which experiments have also been reported very recently.[3] Other systems such as pinned charge-density waves also bear a strong resemblance to disordered flux lattices.[4,5]

In this paper we analyze a realistic model for pinned lattices using a variational replica-field theory. This general approach to the study of the equilibrium impurity-induced deformations of any ordered (crystalline) structure, some aspects of which are discussed in Sec. IV B, was introduced recently for the related problem of a "directed manifold" in a random potential.[6] The results we obtain for certain observable quantities are quite different from those derived using the Larkin model and appear to be in good agreement with available experiments. A Letter describing some of the results explained here has previously appeared.[7]

II. MODEL AND SUMMARY OF THE RESULTS

The model we choose to focus on is the triangular Abrikosov lattice of vortex lines in three dimensions interacting with randomly located impurities. Generalization of our results to other dimensions (in particular to thin films) is not difficult and will be given in Sec. IV B. The situation we want to describe is that of a dislocation-free, "solid" array of flux lines, for which the energy of small distortions is given by the following elastic Hamiltonian H_{elastic} (see, for example, Ref. 8):

$$H_{\text{elastic}}=\frac{1}{2}\int d^2\mathbf{x}\,dz\left[(C_{11}-C_{66})\left[\sum_{\alpha}\partial_\alpha u_\alpha\right]^2+C_{66}\sum_{\alpha,\beta}(\partial_\alpha u_\beta)^2+C_{44}\sum_{\alpha}(\partial_z u_\alpha)^2\right], \tag{1}$$

where α and β are indices denoting the (in plane) x and y directions. $\mathbf{x}$ and z are "internal" coordinates labeling the vortex lines in the xy plane and z direction and coincide with the unperturbed position of these lines. The field $\mathbf{u}(\mathbf{x},z)$ is the displacement in the xy plane of the vortex line from its equilibrium position, and C_{11}, C_{66}, and C_{44} are the bulk, shear, and tilt moduli, which might in general be wave vector dependent.[8] For definiteness, we consider here the three-dimensional problem. The two-dimensional case is obtained by dropping the z integral and setting $C_{44}\equiv 0$.

This elastic Hamiltonian does not describe a possible entangled, liquid phase.[9] The fact that we neglect the presence of dislocations restricts our theory to high-vortex-density regimes where the distance between dislocations is large.[2,3] Recent attempts at including the effects of dislocations can be found in Refs. 10 and 11.

In the original Larkin model, as well as in subsequent elaborations on it,[10,12–14] the presence of impurities is described by a pinning Hamiltonian

$$H_{\mathrm{Larkin}}=\int d^2\mathbf{x}\,dz\,\mathbf{u}(\mathbf{x},z)\cdot\mathbf{f}(\mathbf{x},z)\ , \tag{2}$$

in which each vortex is subject to an *independent random force.* This exactly soluble model is rather unrealistic since two different vortices which wander to the same point in space at different times "feel" *different* pinning forces when they are at that point: The random force is assigned according to the label of the vortex $(\mathbf{x},z)$ and does not depend on its position in space, $\mathbf{r}$. A related consequence of this form of disorder is that there are no metastable configurations of the vortex lines: The minimum of $[H_{\mathrm{elastic}}+H_{\mathrm{Larkin}}](\mathbf{u})$ is unique and can be explicitly constructed for any given set of forces. As we shall see below (and as previously recognized by Feigel'man *et al.*[15]), Larkin's model may be justified at *small length scales,* but fails to describe the correct physics at larger length scales.

In our model we use the same elastic Hamiltonian (1), but take a pinning potential that depends on the position in space of the vortices. For technical reasons only, it is convenient to consider a general model where the pinning potential also depends on the "internal" label $\mathbf{x}$. We take care to include the discrete nature of the vortices: The labels $\mathbf{x}$ build a triangular lattice of spacing a_0. The position of the vortex line labeled $\mathbf{x}$ is $\mathbf{r}(\mathbf{x},z)=\mathbf{x}+\mathbf{u}(\mathbf{x},z)$. We are interested in the case where the displacement field $\mathbf{u}$ is smooth on the level of one lattice spacing $[|\mathbf{u}(\mathbf{x}+a_0\mathbf{1})-\mathbf{u}(\mathbf{x})|\ll a_0]$ so that we can still safely use the continuum model (1) for the elastic energy. The pinning potential is now

$$H_{\mathrm{pin}}=\int dz\sum_{\mathbf{x}}V(\mathbf{r}(\mathbf{x},z),z,\mathbf{x})\ , \tag{3}$$

where V is a Gaussian random pinning potential with zero mean. In the case of a layered superconductor with the field parallel to the layers, an extra pinning term should be added to describe the periodic modulation imposed by the layers. This leads to an interesting phase diagram (see Ref. 16). We assume

$$\overline{V(\mathbf{r},z,\mathbf{x})V(\mathbf{r}',z',\mathbf{x}')}=\frac{U_p^2}{\Delta_z^2}\exp\left[-\frac{(\mathbf{r}-\mathbf{r}')^2}{2\Delta_{xy}^2}-\frac{(z-z')^2}{2\Delta_z^2}\right]f\left[\frac{|\mathbf{x}-\mathbf{x}'|}{a_0}\right]\ , \tag{4}$$

where the overbar denotes a disorder average. U_p^2 is the square of the typical interaction energy between impurities and the vortex core, while Δ_{xy} and Δ_z are the correlation lengths in the plane and z direction, respectively. In what follows, all distances (including the displacements $\mathbf{u}$) will be written in units of a_0 in the xy plane and in units of Δ_z in the z direction. We introduce $C_1\equiv C_{11}a_0^2\Delta_z$, $C_6\equiv C_{66}a_0^2\Delta_z$, and $C_4\equiv C_{44}a_0^4\Delta_z^{-1}$, which have dimensions of an energy, and define $W\equiv(2\pi)^{3/2}U_p^2\Delta_{xy}^2a_0^{-2}$, which has the dimension of an energy squared and is a natural measure of the strength of the disorder.

For convenience, we have also introduced in (4) an "internal" correlation function $f(x)$, which serves as a regularization for some intermediate steps of the computation. This function is supposed to behave as $x^{-\lambda}$ for large x; the physical limit we are interested in is $\lambda\to 0$, since all the vortex lines "see" the *same* pinning potential. It is interesting to note that if $f(x)$ were instead a δ function, our model would become identical to a $d=3$ dimensional directed manifold with $n=2$ transverse components in a random potential (see, e.g., Refs. 6, 15, and 17), where each particle "sees" an *independent* pinning potential. Larkin's random-force Hamiltonian (2), on the other hand, is an approximation of the real pinning potential (3), which is valid only for very short length scales (such that the typical displacements of the vortices are much smaller than both the correlation length of the disorder and the lattice spacing, as we shall see later). In this paper we shall mainly concentrate on the behavior at large length scales, where the random-force approximation is not valid and the existence of many metastable states becomes crucial.

The quantity we shall be most interested in is the in-plane displacement correlation function, defined as

$$\widetilde{B}_{\alpha\beta}(\mathbf{x},z)=\overline{\langle[u(\mathbf{x},z)_\alpha-u(0,0)_\alpha][u(\mathbf{x},z)_\beta-u(0,0)_\beta]\rangle}\ , \tag{5}$$

with $z=0$. $\widetilde{B}$ measures the growth of fluctuations with distance and behaves in general as a power law $\simeq x^{2\nu}$ for large (ν is generally called the "wandering" exponent). It is directly related to the translational density correlation function measured on Bitter patterns (see Sec. V).

Our final results for the three-dimensional case are summarized in Fig. 1. When the correlation length of the potential is small compared with the lattice spacing $(\Delta\equiv\Delta_{xy}/a_0\ll 1)$, we find three different regimes, depending on the order of magnitude of $\widetilde{B}$ (in the following qualitative presentation, we ignore the tensorial nature of $\widetilde{B}$). At short length scales, we recover the results of Larkin and Ovchinnikov:[12,13]

$$\widetilde{B}(x)\sim\Delta^2\left[\frac{x}{\xi_\Delta}\right]\ , \tag{6a}$$

for $1\ll x\ll\xi_\Delta$, with $\xi_\Delta\equiv\Delta^4C_6^{3/2}C_4^{1/2}/U_p^2$. Here and in the rest of the paper, we shall assume that $C_1\gg C_6$, i.e., that shear is much easier than compression, which is thus avoided in low-energy configurations. At intermediate length scales $\xi_\Delta\ll x\ll\xi$, we find the same behavior as for the $n=2$, $d=3$ "random-manifold" problem, for which one has (within the replica variational theory)

$$\widetilde{B}(x)\sim\left[\frac{x}{\xi}\right]^{1/3}\ , \tag{6b}$$

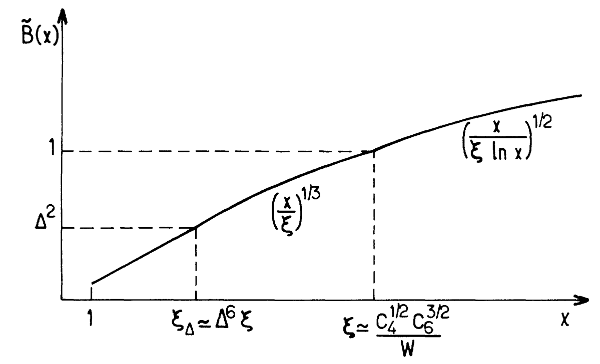

FIG. 1. Schematic plot of the growth of the fluctuations with distance, $\tilde{B}(x)$, showing the three regimes discussed in the text: short-distance regime, where $\tilde{B}(x) \ll \Delta^2$, where Larkin's random-force model is justified; the intermediate-distance regime $1 \gg \tilde{B}(x) \gg \Delta^2$, where the random-manifold exponents are expected; and the asymptotic regime $\tilde{B}(x) \gg 1$, where new exponents prevail.

with now $\xi \equiv C_6^{3/2} C_4^{1/2}/W$. Finally, at large scales $x \gg \xi$, we obtain, in the physical limit $\lambda=0$,

$$\tilde{B}(x) \sim \left[\frac{x}{\xi \ln x} \right]^{1/2} . \tag{6c}$$

In Eqs. (6) the precise numerical prefactors have been omitted, and will be given below [Eqs. (49)–(51)].

Before entering the details of our variational calculation, we shall explain in simple physical terms the meaning of the above results. In the first regime ($x \ll \xi_\Delta$), the displacement u is much smaller than the correlation length of the disordered potential Δ and thus also much smaller than the lattice spacing. Hence one can expand $V(\mathbf{x}+\mathbf{u})$ as $V(\mathbf{x}) + \mathbf{f}(\mathbf{x})\mathbf{u} + \cdots$; we are in a region where Larkin's Hamiltonian can be justified. The short-distance result summarized in Eq. (6a) can be obtained using an Imry-Ma argument (but see also Ref. 12) by minimizing the energy of a given volume $x^2 z$:

$$\delta E \simeq \left[C_6 \left[\frac{u}{x} \right]^2 + C_4 \left[\frac{u}{z} \right]^2 \right] x^2 z - \frac{U_p}{\Delta} u (x^2 z)^{1/2} . \tag{7a}$$

The potential energy is indeed the sum of $x^2 z$ *independent* terms of order $(U_p/\Delta)u$. Minimizing the energy cost, one finds $z \sim (\sqrt{C_4/C_6})x$ and hence the short-distance result

$$B(x) \sim u^2 \sim x \frac{U_p^2}{\Delta^2} C_4^{-1/2} C_6^{-3/2} .$$

In the second (intermediate) regime, the typical displacement is much larger than Δ, but much smaller than the lattice spacing ($=1$). Hence vortices do not wander sufficiently far to "realize" that they are all subject to the same potential. The random potential is effectively independent for each vortex, and the problem is indeed equivalent to an $n=2$, $d=3$ random manifold, for which the variational theory predicts $\nu = \frac{1}{6}$. Finally, in the last regime $x \gg \xi$, the displacement is much larger than the lattice spacing, allowing many favorable configurations, differing by a local translation of the lattice, to exist. A modified Imry-Ma argument can be used to recover the long-distance result given in Eq. (6c). For a random *potential* (rather than a random force), one writes the energy as

$$\delta E \simeq \left[C_6 \left[\frac{u}{x} \right]^2 + C_4 \left[\frac{u}{z} \right]^2 \right] x^2 z - U_p \sqrt{\ln u}\, (x^2 z)^{1/2} , \tag{7b}$$

where the term $\sqrt{\ln u}$ expresses the fact that larger displacements probe the tails of the Gaussian disorder and hence improve slightly the energy gain. From the minimization of the energy cost in Eq. (7b), one indeed finds $\tilde{B}(x) \sim u^2 \sim (x/\xi \ln x)^{1/2}$ (with the correct power of the logarithmic correction).

If the correlation length of the disorder were larger than the lattice spacing, the intermediate random-manifold regime would disappear and one would find, for $x \ll \xi'_\Delta$,

$$\tilde{B}(x) \sim \Delta^2 \left[\frac{x}{\xi'_\Delta} \right] , \tag{8}$$

with $\xi'_\Delta = \Delta^4 \xi = \xi_\Delta / \Delta^2$, which would cross over to the same long-distance behavior as given in Eq. (6c) for $x \gg \xi'_\Delta$.

Finally, while translational order is destroyed, we find that in the absence of dislocations, orientational order remains in three *and* two dimensions, contrary to the findings of Chudnovsky.[13]

III. REPLICA VARIATIONAL THEORY

A. Saddle-point equations

To make analytical progress in computing the free energy F, we first average over the disorder using the replica method.[18] We find an effective replica Hamiltonian

$$H_n = \frac{1}{2} \sum_a \int d^2\mathbf{x}\, dz \left[(C_1 - C_6) \left[\sum_\alpha \partial_\alpha u_\alpha^a \right]^2 + C_6 \sum_{\alpha\beta} (\partial_\alpha u_\beta^a)^2 + C_4 \sum_\alpha (\partial_z u_\alpha^a)^2 \right]$$
$$- \frac{W}{2T} \sum_{ab} \sum_{\mathbf{x}\mathbf{x}'} \int dz\, \delta^{(2)}[\mathbf{x} + \mathbf{u}^a(\mathbf{x},z) - \mathbf{x}' - \mathbf{u}^b(\mathbf{x}',z)] f(|\mathbf{x}-\mathbf{x}'|) , \tag{9}$$

where T is the temperature and a and b are replica indices which run from 1 to n. We have also taken, for simplicity, the limit $\Delta_{xy},\Delta_z \ll a_0$ and replaced the Gaussian correlation function appearing in Eq. (4) by a δ function, which is justifiable if we focus on distances much larger than a_0. The modifications due to a finite correlation length will be considered in Sec. IV C.

Our method is to find the best quadratic approximation of the Hamiltonian (9) in the limit $n=0$. We take, as a trial Hamiltonian (in Fourier space),

$$H_0=\frac{1}{2}\int\frac{d^2\mathbf{q}\,dq_z}{(2\pi)^3}\sum_{ab}\sum_{\alpha\beta}u_\alpha^a(\mathbf{q},q_z)(G^{-1})_{ab}^{\alpha\beta}(\mathbf{q},q_z)u_\beta^b(-\mathbf{q},-q_z) \tag{10}$$

and use the convexity inequality $\tilde{F}(G)\equiv F_0+\langle H_n-H_0\rangle_0\geq F$ to find the best G, minimizing $\tilde{F}(G)$. (The notation $\langle A\rangle_0$ means that we take the average of A over the trial Hamiltonian H_0.)

It is convenient to define

$$B_{\alpha\beta}^{ab}(\mathbf{x})=\langle[u(\mathbf{x},z)_\alpha^a-u(\mathbf{0},z)_\alpha^b][u(\mathbf{x},z)_\beta^a-u(\mathbf{0},z)_\beta^b]\rangle_0 \tag{11}$$

and to decompose $G_{\alpha\beta}^{ab}(\mathbf{q})$ and $B_{\alpha\beta}^{ab}(\mathbf{x})$ into their longitudinal and transverse components in the standard way. For example,

$$G_{\alpha\beta}^{ab}(\mathbf{q},q_z)\equiv\left[\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{q^2}\right]G_T^{ab}(q,q_z)+\frac{q_\alpha q_\beta}{q^2}G_L^{ab}(q,q_z)\ . \tag{12}$$

From their definitions [Eq. (10)], B is connected to G by the equations

$$\begin{aligned}B_L^{ab}(x)=T\int\frac{d^2\mathbf{q}\,dq_z}{(2\pi)^3}\{&[G_L^{aa}(q,q_z)+G_L^{bb}(q,q_z)-2G_L^{ab}(q,q_z)\cos(\mathbf{q}\cdot\mathbf{x})]\cos^2\phi\\&+[G_T^{aa}(q,q_z)+G_T^{bb}(q,q_z)-2G_T^{ab}(q,q_z)\cos(\mathbf{q}\cdot\mathbf{x})]\sin^2\phi\}\ .\end{aligned} \tag{13}$$

($x\equiv|\mathbf{x}|$, $q\equiv|\mathbf{q}|$, and ϕ is the angle between $\mathbf{q}$ and $\mathbf{x}$.) The corresponding formulas for the transverse components B_T are obtained by inverting the roles of $\cos^2\phi$ and $\sin^2\phi$. Using the fact that, within the variational ansatz, $\{u(\mathbf{x},z)_\alpha^a\}$ are Gaussian variables, the trial free energy $\tilde{F}(G)$ is easily found to be

$$\begin{aligned}\frac{\tilde{F}(G)}{V}=\frac{T}{2}\int\frac{d^2\mathbf{q}\,dq_z}{(2\pi)^3}\sum_a\big(&C_1q^2G_L^{aa}(q,q_z)+C_6q^2G_T^{aa}(q,q_z)+C_4q_z^2[G_L^{aa}(q,q_z)+G_T^{aa}(q,q_z)]\\&-\{\ln[TG_T(q,q_z)]\}^{aa}-\{\ln[TG_L(q,q_z)]\}^{aa}\big)\\&-\frac{W}{2T}\sum_{a,b}\sum_{\mathbf{x}}\frac{1}{(2\pi)[B_L^{ab}(x)B_T^{ab}(x)]^{1/2}}e^{-x^2/2B_L^{ab}(x)}f(x)\ ,\end{aligned} \tag{14}$$

where V is the volume of the system.

G is chosen to minimize $\tilde{F}(G)$ and thus satisfies

$$\frac{\partial\tilde{F}(G)}{\partial G_{ab}^{\alpha\beta}(\mathbf{q},q_z)}\equiv 0\ , \tag{15}$$

which leads to the saddle-point equations

$$\begin{aligned}[G_L^{-1}]_{aa}(q,q_z)=C_1q^2+C_4q_z^2&+\frac{W}{2\pi T}\sum_{\mathbf{x}}\frac{e^{-x^2/2B_L^{aa}(x)}f(x)}{[B_L^{aa}(x)B_T^{aa}(x)]^{1/2}}[1-\cos(\mathbf{q}\cdot\mathbf{x})]\left[\frac{\sin^2\phi}{B_T^{aa}(x)}+\frac{\cos^2\phi}{B_L^{aa}(x)}\left[1-\frac{x^2}{B_L^{aa}(x)}\right]\right]\\&+\frac{W}{2\pi T}\sum_{b(\neq a)}\sum_{\mathbf{x}}\frac{e^{-x^2/2B_L^{ab}(x)}f(x)}{[B_L^{ab}(x)B_T^{ab}(x)]^{1/2}}\left[\frac{\sin^2\phi}{B_T^{ab}(x)}+\frac{\cos^2\phi}{B_L^{ab}(x)}\left[1-\frac{x^2}{B_L^{ab}(x)}\right]\right]\end{aligned} \tag{16a}$$

and

$$[G_L^{-1}]_{a\neq b}(q,q_z)=-\frac{W}{2\pi T}\sum_{\mathbf{x}}\frac{e^{-x^2/2B_L^{ab}(x)}f(x)}{[B_L^{ab}(x)B_T^{ab}(x)]^{1/2}}\cos(\mathbf{q}\cdot\mathbf{x})\left[\frac{\sin^2\phi}{B_T^{ab}(x)}+\frac{\cos^2\phi}{B_L^{ab}(x)}\left[1-\frac{x^2}{B_L^{ab}(x)}\right]\right]\ . \tag{16b}$$

Similar equations are obtained for the transverse component G_T, with C_6 replacing C_1 and by inverting the roles of $\cos^2\phi$ and $\sin^2\phi$. As always in the replica method, one must take eventually the limit $n\to\infty$.

The disorder average of the fluctuations is determined from the diagonal components of B; for example,

$$\tilde{B}_L(x)\equiv B_L^{aa}(x)=\overline{\left\langle\left[[\mathbf{u}(\mathbf{x},z)-\mathbf{u}(\mathbf{0},z)]\cdot\frac{\mathbf{x}}{x}\right]^2\right\rangle}\ , \tag{17}$$

while the thermal part of the fluctuations is governed by the "connected" part G^c of G, defined as

$$G^c_{L,T}(q,q_z)=G^{aa}_{L,T}(q,q_z)+\sum_{a\neq b} G^{ab}_{L,T}(q,q_z)\ , \tag{18}$$

which satisfies the relation

$$G^c_{L,T}(q,q_z)=\frac{1}{(G^{-1}_{L,T})^c(q,q_z)}\ . \tag{19}$$

(The notation $G_{L,T}$ is just an abbreviation for two equations, one for G_L and one for G_T.) From Eqs. (16) one readily sees that, in general,

$$(G^{-1}_{L,T})^c(q\to 0,q_z)=C_{L,T}q^2+C_4q_z^2\ , \tag{20}$$

thereby defining the renormalized bulk and shear moduli.

B. Replica-symmetric solution

The simplest solution to the saddle-point equations (16) is the replica-symmetric one, obtained by setting $\tilde{G}\equiv G^{aa}$ and $G\equiv G^{a\neq b}$. To invert the corresponding 0×0 matrices, one uses Eq. (19) for $G^c\equiv\tilde{G}-G$ and $(G^{-1})^{a\neq b}=-G/(G^c)^2$. From Eq. (16b) one finds that, in the small-q limit,

$$\tilde{G}_{L,T}\simeq G_{L,T}\simeq\frac{g_{L,T}}{(C_{L,T}q^2+C_4q_z^2)^2}\ , \tag{21}$$

where $g_{L,T}$ are constants. It is easy to see, using Eq. (13), that this q^{-4} divergence induces a linear growth of $\tilde{B}(x)$ with distance: $\tilde{B}(x)\simeq(x/\xi_{\rm RS})^{2\nu}$, with $\nu=\frac{1}{2}$ [and more generally $\nu=(4-d)/2$]. This is precisely the result found in Refs. 12 and 13, using the Larkin model. However, the correlation length $\xi_{\rm RS}$ which we obtain in this replica-symmetric approach is proportional to $(\sqrt{C_L/C_4})T^2/W$ and thus goes to zero at $T=0$, which is clearly unphysical (fluctuations would diverge on all length scales). This solution must therefore be discarded. In any case, the instability of this solution to replica-symmetry breaking can be demonstrated along the lines of Ref. 6.

C. Replica-symmetry broken solution

Following the work of Ref. 6, we look for a replica-symmetry-broken solution. Before entering the technical aspects of this procedure, let us recall, following Ref. 6, its physical interpretation within the present variational approach. For graphical convenience we shall imagine that the phase space of the problem is one dimensional; the energy landscape can be, very schematically, of two types. In Fig. 2(a) is represented a single-well potential and the best Gaussian approximation to the Boltzmann weight, which represents a reasonable approximation. For a disordered potential such as the one drawn in Fig. 2(b), many nearly degenerate minima exist. The description of the system with a unique Gaussian is clearly inadequate—the weight is scattered among well-separated minima. The replica-symmetric solution to the variational equations is an attempt to describe a given disordered sample with a unique Gaussian centered

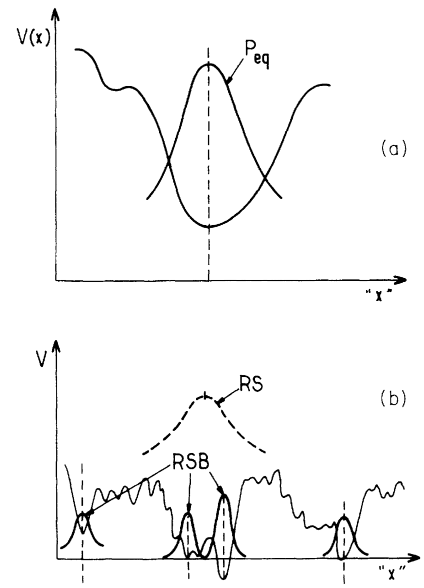

FIG. 2. Two different types of energy landscapes: (a) single-well potential, for which a Gaussian approximation is reasonable, and (b) the situation where many degenerate minima exist, and the description of the Boltzmann weight by a unique Gaussian (the center of which is optimized) is clearly very bad (replica-symmetric solution).

around a sample-dependent point. The position of this point for different samples is itself distributed according to a Gaussian. The two variational parameters (corresponding to the diagonal element $\tilde{G}$ and to the unique off-diagonal element G introduced above) are thus the width of the two Gaussians: one for the thermal fluctuations inside a given sample, the other for the sample to sample fluctuations of the average position.

As detailed in Ref. [6], the replica-symmetry broken solution permits a much more versatile description of the system. The Boltzmann weight for a given sample is approximated by an (infinite) sum of Gaussians, the height, width, and position of which are parametrized by a whole *function* on the interval [0,1]. The variational calculation is used to determine this function. Although the construction implied by Parisi's breaking scheme cannot reproduce all distributions of Boltzmann weights, it was shown for the case of random manifolds that the main physical features of the problem were correctly caught by this ansatz.

Technically, $G^{ab}(q,q_z)$ is chosen to be a hierarchical matrix, which, in the limit $n=0$, is parametrized by a function $G(q,q_z,v)$, where v is a continuous variable $0\le v\le 1$. The hierarchical organization of the matrix is in correspondence with a hierarchical organization of low-lying configurations. From the physical interpretation [see, e.g., Ref. 6, Eq. (6.8)], T/v can be thought of as a free-energy difference between two low-lying configurations at a level of the hierarchy characterized by the number v. These (0×0) hierarchical matrices obey a well-defined algebra; we will need, in particular, the rule for inverting such a matrix, which is summarized for completeness in Appendix A.

The long-distance behavior is dominated by the regime $q \ll 1$, $v \ll 1$. The scaling solution of the stationarity equations (16), which we shall describe below, has different behaviors depending on the ranges of q and v with respect to some crossover values v^* and $q^*_{L,T}$ (there are different momentum crossovers for G_L and G_T), given by

$$v^* = \frac{WT}{C_4 C_T^2}, \quad q^*_{L,T} = \frac{W}{(C_4 C_T^2 C_{L,T})^{1/2}} . \qquad (22)$$

Our study is restricted to the case of low temperature and weak disorder (small W) for which $q^*_{L,T} \ll 1$ and $v^* \ll 1$. Furthermore, we shall keep, for simplicity, to the physically relevant case where $C_T \ll C_L$. The solution we have found for each propagator $G_L(q,q_z,v)$ and $G_T(q,q_z,v)$ has two main scaling regimes (Fig. 3). Regime A corresponds to $v \ll v^*$ or $q \ll q^*_{L,T}$. We shall call it the long-distance regime since it will turn out to dominate the long-distance correlations of flux-line displacements. Regime B, the intermediate-distance regime, corresponds to $1 \gg v \gg v^*$ and $1 \gg q \gg q^*_{L,T}$. In regime A we have found a solution of the type

$$\int_{-\infty}^{+\infty} dq_z G_{L,T}(q,q_z,v) = \frac{N}{T(q_T^*)^2} \left[\frac{q^*_{L,T}}{q} \right]^{2+2\nu} h_\omega \left[N \frac{v/v^*}{(q/q^*_{L,T})^\omega} \right] , \qquad (23)$$

where

$$h_\omega(t) = \pi \left[\frac{1}{t} \{ 1-(1+t^{2/\omega})^{-1/2} \} + \int_0^t \frac{dv}{v^2} \{ 1-(1+v^{2/\omega})^{-1/2} \} \right] \qquad (24)$$

and N is a pure number, which we shall determine together with the exponents ν and ω (ν, ω, and N are equal for the two propagators G_L and G_T). A similar solution is found in regime B, with a different set of exponents and constant: ν',ω',N'. The exponent ν governs the fluctuations of vortex displacements, while ω turns out to be the "energy" exponent, governing the growth of free-energy fluctuations with distance. Indeed, as T/v is a free-energy difference ΔF between two low-lying configurations, the scaling variable v/q^ω means that for a given length scale $L = q^{-1}$ one should compare ΔF to L^ω (ω is also called θ by Fisher and Huse[19]).

As for the diagonal part of the propagator, $\tilde{G}(q,q_z)$, the solution behaves as

$$\int_{-\infty}^{+\infty} dq_z \tilde{G}_{L,T}(q,q_z) = \frac{N}{T(q_T^*)^2} \left[\frac{q^*_{L,T}}{q} \right]^{2+2\nu} h_\omega(\infty) , \qquad (25)$$

for $q \ll q^*_{L,T}$, and a similar expression with ν, ω, and N replaced by ν', ω', and N' for $q \gg q^*_{L,T}$.

To establish that Eqs. (23)–(25) give a solution to the saddle-point equations and to determine the exponents $\nu^{(\prime)}$ and $\omega^{(\prime)}$ and numbers $N^{(\prime)}$, we shall proceed as follows. From the saddle-point equation (16b), one determines the off-diagonal part of the inverse of G, which we will denote as $-\Sigma$ (Σ is a "self-energy"), in terms of B. With the help of Eq. (20) giving the connected part of G in the small-q limit, G can be constructed from Σ using the inversion rule mentioned above. Using Eqs. (23), a self-consistent equation for G is obtained. Let us proceed.

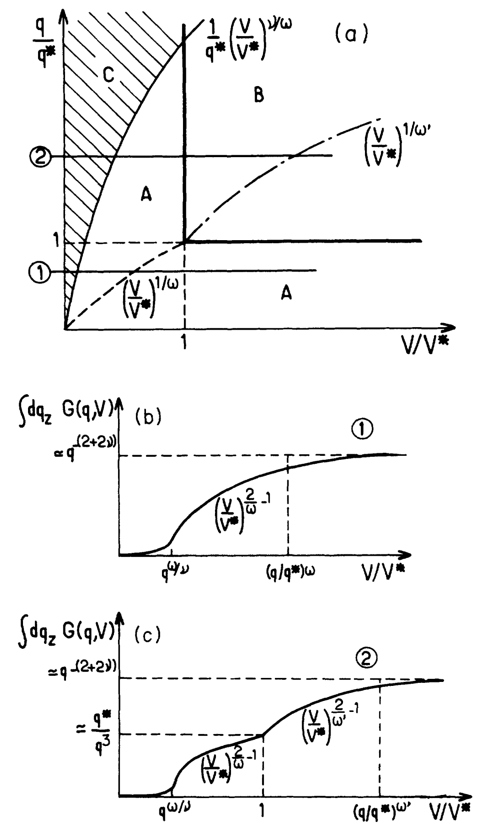

FIG. 3. (a) Off-diagonal propagator (integrated over q_z) is a function of the transverse momentum q and the replica parameter v. In the limit $q \ll 1$, $v \ll 1$, there are three main scaling behaviors of the propagator [see Eq. (23)]. In the long-distance regime A, it is characterized by the exponents ν and ω given by Eq. (34). In the intermediate-distance regime B ($q \gg q^*$, $v \gg v^*$), the exponents ν' and ω' are given by Eq. (46). In regime C the propagator is exponentially small in $q(v^*/v)^{\nu/\omega}$. Also shown are the crossover lines $q/q^* = (v/v^*)^{1/\omega}$ (dashed line) and $q/q^* = (v/v^*)^{1/\omega'}$ (dot-dashed line). Note that the propagator integrated over q (i.e., the one governing the properties in the z direction) also depends on the value of $q_z\sqrt{C_4/C_6}$. (b) and (c) Shape of $\int dq_z G(q,q_z,v)$ for a fixed q, as a function of v: (b) $q < q^*$ and (c) $q > q^*$.

1. Small-v regime $v \ll v^*$

From the assumed form for the propagator given in Eq. (23) and the definition (13), one finds that B is (for $C_T \ll C_L$, i.e., $q_L^* \ll q_T^*$)

$$B_{L,T}(x,v) \simeq \left[\frac{v}{v^*} \right]^{-2\nu/\omega} b_{L,T}[q^*_{L,T}(v/v^*)^{1/\omega} x] , \quad x \gg 1 , \qquad (26)$$

with

$$b_L(r)=2N\int\frac{d^2k}{(2\pi)^3}\frac{1}{k^{2+2\nu}}\times\left[h_\omega(\infty)-h_\omega\left[\frac{N}{k^\omega}\right]\cos(kr\cos\phi)\right]\sin^2(\phi) \tag{27}$$

(ϕ is the angle between the integration vector $\mathbf{k}$ and one axis) and $b_T(r)$ is obtained from $b_L(r)$ by changing $\sin^2(\phi)$ into $\cos^2(\phi)$. We suppose that v is so small that $B_{L,T}(x=1,v)\gg 1$. This is true for v much smaller than

$$v_c=v^*\{b_{L,T}[q^*_{L,T}(v_c/v^*)^{1/\omega}]\}^{\omega/2\nu}\ .$$

We shall find that v_c is of order v^* and

$$b_{L,T}[q^*_{L,T}(v_c/v^*)^{1/\omega}]\simeq b_{L,T}(0)\equiv b_0\ .$$

In this regime the argument of the exponential in Eq. (16) is small in magnitude when x is not too large. Hence a large number of terms will contribute to the sum in the right-hand side of (16b), which we approximate by an integral. The exponential damping starts to be important when

$$x^2\simeq x_c^2\equiv(v/v^*)^{-2\nu/\omega}b_{L,T}[q^*_{L,T}(v/v^*)^{1/\omega}x_c]\ .$$

The condition $(v/v^*)^{1/\omega}q^*_{L,T}x_c\ll 1$ is automatically satisfied when $v\ll v^*$ since $\nu<1$. Hence, for *all* x contributing to the integral (16b), one may write

$$b_{L,T}[q^*_{L,T}(v/v^*)^{1/\omega}x]=b_{L,T}(0)=b_0$$

and

$$B_{L,T}(x,v)\simeq(v/v_c)^{-(2\nu/\omega)}\ ,$$

where

$$v_c=v^*(b_0)^{\omega/2\nu}\ . \tag{28}$$

Upon the change of variables $\mathbf{z}=\mathbf{x}(v/v_c)^{\nu/\omega}$, one finds

$$\Sigma_{L,T}(q,v)=\frac{W}{T}\left[\frac{v}{v_c}\right]^{(2+\lambda)\nu/\omega}\sigma_{L,T}\left[q\left[\frac{v}{v_c}\right]^{-\nu/\omega}\right]\ , \tag{29}$$

where

$$\sigma_L(t)\equiv\frac{2}{\sqrt{3}}\int\frac{d^2z}{2\pi}e^{-z^2/2}[(1-z^2)\cos^2\phi+\sin^2\phi]z^{-\lambda}\times\cos(tz\cos\phi) \tag{30}$$

and similarly for σ_T by inverting the roles of $\cos^2\phi$ and $\sin^2\phi$. Note that $\sigma_L(t)$ goes to a constant for small t and decays quickly to zero when $t\to\infty$.

In order to compute the propagator $G_{L,T}(q,q_z,v)$, we must now invert $\Sigma_{L,T}(q,v)$ in the sense of hierarchical matrices. As recalled in Appendix A, this inversion requires the knowledge of the "connected" propagator $G^c_{L,T}$, which has been derived in Eq. (20), and of the "bracket" transform (defined in Appendix A) of $\Sigma_{L,T}$, which is equal to

$$[\Sigma_{L,T}](q,v)=\frac{W}{T}v\left[\frac{v}{v_c}\right]^{(2+\lambda)\nu/\omega}\hat\sigma_{L,T}\left[q\left[\frac{v}{v_c}\right]^{-\nu/\omega}\right]\ , \tag{31}$$

where

$$\hat\sigma_{L,T}(t)=\sigma_{L,T}(t)-\int_0^1 dw\, w^{(2+\lambda)\nu/\omega}\sigma_{L,T}(tw^{-\nu/\omega})\ . \tag{32}$$

The propagator $G_{L,T}(q,q_z,v)$ can now be read from Appendix A, and the q_z integration gives

$$\int dq_z G_{L,T}(q,q_z,v)=\frac{\pi}{q\sqrt{C_4C_{L,T}}}\left[\frac{1}{v}\left[1-\frac{1}{\{1+[\Sigma_{L,T}](v)/C_{L,T}q^2\}^{1/2}}\right]+\int_0^v\frac{dw}{w^2}\left[1-\frac{1}{\{1+[\Sigma_{L,T}](w)/C_{L,T}q^2\}^{1/2}}\right]\right]\ . \tag{33}$$

Let us now focus on the regime $(q,v)\to 0$ with a fixed ratio v/q^ω. For $\nu<1$, $q(v/v_c)^{-\nu/\omega}$ goes to zero in this limit, and $\hat\sigma_{L,T}$ can be replaced by a constant $\hat\sigma(0)$, identical for L and T components. A look at Eq. (33) then shows that in this regime, $\int dq_z G(q,q_z,v)$ is indeed of the form assumed in Eq. (23) upon the following identification: $2=\omega+(2+\lambda)\nu$, $\omega=1+2\nu$, and $N^{2/\omega}=(b_0)^{w/2\nu-1/\nu}\hat\sigma(0)$. $\hat\sigma(0)$ and b_0 are computed in Appendix B. Our final result is, therefore, in this long-distance regime,

$$\nu=\frac{1}{4+\lambda},\quad \omega=\frac{6+\lambda}{4+\lambda}\ ,$$
$$N=\left[\frac{4+\lambda}{8\pi}\right]^{-(2+\lambda)/(4+\lambda)}\left[\frac{1}{2\sqrt{3}}\frac{2+\lambda}{4+\lambda}\lambda\Gamma\left[1-\frac{\lambda}{2}\right]2^{-\lambda/2}\right]^{2/(4+\lambda)}\ . \tag{34}$$

Note that the relation $\omega=1+2\nu$ relating the energy and wandering exponents can, in fact, be obtained by a simple scaling of the elastic terms of the Hamiltonian. The self-consistency of the assumption for the diagonal part of the propagator given in Eq. (25) will be checked after the study of the $v\gg v^*$ regime.

Before turning to this $v\gg v^*$ situation, let us note that the only additional hypothesis we used to derive (23) in the regime $v\ll v^*$ is the replacement of $q(v/v_c)^{-\nu/\omega}$ by 0 in (32). This hypothesis is wrong when

$q > q_b = (v/v_c)^{\nu/\omega}$, and this defines a small region (C in Fig. 1), where it turns out that $[\Sigma]_{L,T}$ and thus $G_{L,T}$ are very small. (They go to zero exponentially when one departs from the boundary line $q = q_b$.) We shall not need to compute $G_{L,T}$ more precisely in this region since it will not give any contribution to the physical quantities.

2. Intermediate-v regime

In this case the argument of the exponential appearing in (16b) is already large when $x = 1$. We thus only retain, in the sum in the right-hand side of (16b), the very first term $x = 0$. Hence

$$\Sigma_{L,T}(v) = \frac{W}{2\pi T} \frac{1}{B_{L,T}(0,v)^2} . \tag{35}$$

The saddle-point equations are thus of the same form, in this regime, as those obtained for the random-manifold problem in Ref. 6.

We shall see below that $B_{L,T}(0,v)$ takes the form

$$B_L(0,v) = B_T(0,v) = \left[\frac{v}{v^*} \right]^{-2\nu'/\omega'} b_0' , \tag{36}$$

which implies

$$\Sigma_{L,T}(v) = \frac{W}{2\pi T b_0'^2} \left[\frac{v}{v^*} \right]^{4\nu'/\omega'} . \tag{37}$$

In order to compute the propagator, we need the bracket transform $[\Sigma]_{L,T}(q,v)$, which also receives some contribution from the regime $v < v^*$. However, it is easy to see that this contribution is negligible, so that

$$[\Sigma_{L,T}](v) \simeq \frac{W}{2\pi T b_0'^2} \frac{4\nu'}{\omega' + 4\nu'} v \left[\frac{v}{v^*} \right]^{4\nu'/\omega'} . \tag{38}$$

The inversion formula (33) giving $\int dq_z G_{L,T}(q,q_z,v)$ still holds. Again, this formula involves an integral over w between 0 and v, which we subdivide into a contribution from the region $[0,v^*]$ and another contribution from the region $[v^*,v]$. Thus we write

$$\begin{aligned} &\int dq_z G_{L,T}(q,q_z,v) = G^<_{L,T}(q) + G^>_{L,T}(q,v) , \\ &G^<_{L,T}(q) \equiv \frac{\pi}{q\sqrt{C_4 C_{L,T}}} \int_0^{v^*} \frac{dw}{w^2} \left[1 - \frac{1}{\{1 + [\Sigma_{L,T}](w)/C_{L,T} q^2\}^{1/2}} \right] , \\ &G^>_{L,T}(q,v) \equiv \frac{\pi}{q\sqrt{C_4 C_{L,T}}} \left[\frac{1}{v} \left[1 - \frac{1}{\{1 + [\Sigma_{L,T}](v)/C_{L,T} q^2\}^{1/2}} \right] + \int_{v^*}^{v} \frac{dw}{w^2} \left[1 - \frac{1}{\{1 + [\Sigma_{L,T}](w)/C_{L,T} q^2\}^{1/2}} \right] \right] . \end{aligned} \tag{39}$$

Let us first consider the case where $q \gg q^*_{L,T}$. Then it is clear that $G^>_{L,T}(q,v)$ takes back the assumed scaling form (23), provided that

$$2 = \omega' + 4\nu', \quad \omega' = 1 + 2\nu', \quad (N')^{2/\omega'} = \frac{1}{2\pi} \frac{4\nu'}{\omega' + 4\nu'} \left[\frac{1}{b_0'} \right]^2 . \tag{40}$$

In this same regime $q \gg q^*_{L,T}$, $G^<_{L,T}$ takes the form

$$G^<_{L,T}(q) = \frac{N^{2/\omega}}{T(q^*_T)^2} \left[\frac{q^*_{L,T}}{q} \right]^3 \frac{\omega}{2(2-\omega)} \tag{41}$$

and one can check that this is indeed much smaller than $G^>_{L,T}$, which proves the validity of the solution with parameters ν', ω', N' in the regime $q \gg q^*_{L,T}$, $v \gg v^*$.

When $q \ll q^*_{L,T}$ (but still $v \gg v^*$), one finds, on the contrary, that $G^> \ll G^<$. This regime is still dominated by a scaling form (23), but with the parameters ν, ω, and N of the $v \ll v^*$ regime. One easily checks that within this hypothesis one gets $G^> \simeq \pi/(v^* q\sqrt{C_4 C_{L,T}})$, which is much smaller than $G^<$, so that, in this regime,

$$\int dq_z G_{L,T}(q,q_z,v) = \frac{N}{T(q^*_T)^2} \left[\frac{q^*_{L,T}}{q} \right]^{2+2\nu} h_\omega(\infty) . \tag{42}$$

The last missing step is to compute the value of N' from (40). We need b_0', which is derived from

$$\begin{aligned} &B(0,v) = \mathcal{B}_L(v) + \mathcal{B}_T(v) , \\ &\mathcal{B}_{L,T}(v) \equiv T \int \frac{d^2 q}{(2\pi)^3} \int dq_z [\tilde{G}_{L,T}(q,q_z) - G_{L,T}(q,q_z,v)] . \end{aligned} \tag{43}$$

It is convenient to use the sum rule (A5) of Appendix A to express $\tilde{G}_{L,T}(q,q_z) - G_{L,T}(q,q_z,v)$. Performing the q_z integration, we get

$$\mathcal{B}_{L,T}(v) = \frac{T}{2\sqrt{C_4 C_{L,T}}} \int_0^\infty \frac{dq}{2\pi} \left[\frac{1}{v\{1 + [\Sigma_{L,T}](v)/C_{L,T} q^2\}^{1/2}} - \int_v^1 \frac{dw}{w^2} \frac{1}{\{1 + [\Sigma_{L,T}](w)/C_{L,T} q^2\}^{1/2}} \right] . \tag{44}$$

This q integral must be divided into two pieces $q < q^*_{L,T}$ and $q > q^*_{L,T}$ since $[\Sigma]_{L,T}$ takes different forms in these regions. One finds that the small-q integral is much smaller than the other one, so that

$$\mathcal{B}_{L,T}(v) = \frac{T}{2\sqrt{C_4 C_{L,T}}}(N')^{1/\omega'}\frac{1}{v^*}\left[\frac{v}{v^*}\right]^{-2\nu'/\omega'} q^*_{L,T}\int_0^\infty \frac{dk}{2\pi}\left[\frac{1}{(1+k^{-2})^{1/2}} - \int_1^\infty \frac{du}{u^2}\,\frac{1}{(1+k^{-2}u^{2/\omega'})^{1/2}}\right] . \quad (45)$$

Denoting by $K(\omega')$ this last integral (computed in Appendix B), this gives back the form (36) for $B(0,v)$, with $b'_0 = (1/2)N'^{(1/\omega')}K(\omega')$. From the self-consistency condition (40), we can now write the full set of parameters which characterize the scaling solution for $q \gg q^*_{L,T}$, $v \gg v^*$:

$$\nu' = \tfrac{1}{6}, \quad \omega' = \tfrac{4}{3}, \quad N' = \tfrac{2}{3}\pi^{1/3} . \quad (46)$$

Finally, we also need to check the self-consistency of the diagonal part of the propagator, $\int dq_z \tilde{G}_{L,T}(q,q_z)$ in (25). Appendix A provides the sum rule

$$\int_{-\infty}^{+\infty} dq_z \tilde{G}_{L,T}(q,q_z) = \frac{\pi}{\sqrt{C_4}}\left[\frac{1}{(C_{L,T}q^2)^{1/2}} + \int_0^1 \frac{dv}{v^2}\left[\frac{1}{(C_{L,T}q^2)^{1/2}} - \frac{1}{\{C_{L,T}q^2 + [\Sigma_{L,T}](v)\}^{1/2}}\right]\right] . \quad (47)$$

We cut the v integral into two pieces $v < v^*$ and $v > v^*$. It may be seen that the first piece dominates when $q \ll q^*_{L,T}$, while the second one gives the dominant contribution for $q \gg q^*_{L,T}$. This gives back the assumed form (25), with exponents ν, ω or ν', ω' depending on the regime.

3. *Prediction for the average fluctuations*

We are now in position to obtain the physical quantities of interest, namely, the displacements of the vortex positions in the transverse plane. These are measured by the displacement correlation $\tilde{B}_{\alpha\beta}(\mathbf{x})$ between vortex $\mathbf{x}$ and vortex $\mathbf{0}$, defined in (5). This correlation is related to the diagonal propagator $\tilde{G}_{L,T}(q,q_z)$ through Eq. (13) (with $a=b$). We just need to substitute the solution (25) for the propagator. In the limit $C_T \ll C_L$, the leading contribution to (13) comes from the transverse propagator $\tilde{G}_T$. The q integral in (13) is dominated either by the regime $q \ll q^*_T$ (if $q^*_T|\mathbf{x}| \gg 1$) or by the regime $q \gg q^*_T$ (if $q^*_T|\mathbf{x}| \ll 1$). We thus find two different scaling behaviors for the correlation functions $\tilde{B}_{L,T}(\mathbf{x})$, depending on the relative values of $|\mathbf{x}|$ and the correlation length ξ:

$$\xi = \frac{1}{q^*_T} = \frac{(C_4 C_T^3)^{1/2}}{W} . \quad (48)$$

The result is (see Fig. 1)

$$\tilde{B}_{L,T}(\mathbf{x}) = \begin{cases} I_{L,T}\left[\dfrac{x}{\xi}\right]^{2\nu} & \text{if } x \gg \xi , \\ I'_{L,T}\left[\dfrac{x}{\xi}\right]^{2\nu'} & \text{if } 1 \ll x \ll \xi , \end{cases} \quad (49)$$

where $I^{(\prime)}_{L,T}$ are pure numbers, which are equal to

$$I_{L,T} = 2Nh_\omega(\infty)N_{s,c}(\nu) . \quad (50)$$

[The integrals $N_{s,c}(\nu)$, which come from the evaluation of (13), are defined and computed in Appendix B.) The values of $I'_{L,T}$ are given by the same expression as (50) with ω, ν, N replaced by ω', ν', and N'. The final values of these constants are

$$I_L \simeq \frac{(2\lambda)^{1/2}}{3^{1/4}5} \quad (\lambda \to 0), \quad I'_L = \frac{3}{7\pi^{2/3}}\Gamma\left[\frac{2}{3}\right] ,$$
$$I_T = (2\nu+1)I_L = \frac{6+\lambda}{4+\lambda}I_L , \quad (51)$$
$$I'_T = (2\nu'+1)I'_L = \frac{4}{3}I'_L .$$

For $x \sim \xi$, $\tilde{B}_{L,T}(x) \sim 1$, which means that the typical relative displacement is of order 1. The change of exponents at this scale reflects the fact, anticipated in Sec. II, that vortices start realizing that they are all standing in the same landscape when they can be moved by at least one lattice spacing. Note that the ratio of $\tilde{B}_T$ to $\tilde{B}_L$ is fixed in each regime and is equal to $2\nu+1$, where ν is the effective exponent of the regime under consideration. Three more remarks should be made.

(i) The whole computation, based on linear elasticity, only makes sense if $\tilde{B}(1) \ll 1$, i.e., $\xi \gg 1$.

(ii) The growth of the fluctuations in the z direction (along a vortex), defined by Eq. (5) with $\mathbf{x}=0$, depends on the combination $\sqrt{(C_6/C_4)}z$. If $\sqrt{(C_6/C_4)}z \ll 1$, the interactions between the vortices (i.e., C_6) can be neglected and the vortices behave as *isolated directed polymers* in a random environment, for which $d=1$ and $n=2$. Within our approximation we thus find $\tilde{B}(z) \simeq z^{2\nu_{DP}}$, with $\nu_{DP} = \frac{1}{2}$. (Numerical simulations and the conjecture of Refs. 15 and 17 suggest rather $\nu_{DP} \simeq \frac{3}{5}$.) In the asymptotic regime $(\sqrt{C_6/C_4})z \gg 1$, $\tilde{B}(z)$ is simply obtained from $\tilde{B}(x)$ by substituting x with $(\sqrt{C_6/C_4})z$ (up to a numerical prefactor).

(iii) The reduced structure factor $S(q) = \int_{-\infty}^{+\infty} dq_z S(q,q_z)$ can be obtained by noticing that the vortex-density fluctuation $\delta\rho$ is related to $\mathbf{u}$ through $\delta\rho = -(\sum_\alpha \partial_\alpha u_\alpha)$. Hence the deviation of $S(q)$ from a usual Bragg-peak structure is given by $\delta S(q) \simeq q^2 \int_{-\infty}^{+\infty} dq_z \tilde{G}(q,q_z)$. In particular, $S(q) \simeq q^{-2\nu}$ $(q \to 0)$.

4. "Connected" equation and the renormalized elastic moduli

We are now in position, using Eqs. (16), (18), and (20), to estimate the shift of elastic moduli induced by the presence of the random potential. Let us first note that this shift is strictly zero for the "random-manifold" case, where the disorder is uncorrelated from vortex to vortex. This can be proved along the lines of Refs. 20–22, but can also directly be seen in Eqs. (16) and (18) using $f(x)=\delta(x)$, since

$$C_L=C_1+\frac{W}{2\pi T}\int_0^1 dv\sum_{\mathbf{x}\neq 0}\frac{e^{-x^2/2\tilde{B}_L(x)}f(x)}{\sqrt{\tilde{B}_L(x)\tilde{B}_T(x)}}[1-\cos(\mathbf{q}\cdot\mathbf{x})]\left[\frac{\sin^2\phi}{\tilde{B}_T(x)}+\frac{\cos^2\phi}{\tilde{B}_L(x)}\left[1-\frac{x^2}{\tilde{B}_L(x)}\right]\right]$$

$$-\frac{W}{2\pi T}\int_0^1 dv\sum_{\mathbf{x}\neq 0}\frac{e^{-x^2/2B_L(x,v)}f(x)}{\sqrt{B_L(x,v)B_T(x,v)}}\left[\frac{\sin^2\phi}{B_T(x,v)}+\frac{\cos^2\phi}{B_L(x,v)}\left[1-\frac{x^2}{B_L(x,v)}\right]\right] \tag{52}$$

(and similarly for C_T). In the opposite case $f(x)\equiv 1$, one cuts the v integral into two pieces (from 0 to v^* and from v^* to 1). The first piece may be seen to be *positive*, of order $(W/T)v^*\simeq C_T/\xi^2$. The second piece may be shown to be of order $\exp[-(\xi^{2\nu})]\ll 1$, using the fact that, for $v>v^*$, $B(x,v)\to\tilde{B}(x)$ as $T\to 0$. Hence, in the limit $\xi\gg 1$, we find

$$\frac{\Delta C_1}{C_1}\propto\frac{\Delta C_6}{C_6}\propto+\frac{C_6}{\xi^2}\ , \tag{53}$$

which shows that disorder tends to slightly *stiffen* the lattice.

IV. RESULTS

A. Predictions in $d=3$ and in the physical limit $\lambda=0$

1. Limit $\lambda=0$

We now examine the physical limit where the random potential is the same for all the vortices. Thus the correlation function f defined by Eq. (4) does not depend on $x-x'$ and hence $\lambda=0$. This limit is slightly subtle because N vanishes: $N\sim\sqrt{\lambda}$ [see Eq. (34)]. But then the prefactor of $\tilde{B}(x)$ [see Eqs. (49) and (51)] also goes to zero, while the exponent $\nu=1/(4+\lambda)$ increases. What happens in this limit is that the different quantities do not have pure power-law behaviors, but that logarithmic corrections come into play. These logarithmic terms can be obtained through the following arguments. From physical intuition it is clear that the more correlated the potential is, the larger the fluctuations, since a favorable region is of interest to *all* vortices. Hence $\tilde{B}(x,\lambda)$ should be, for a fixed x, increasing when λ decreases. From (49) we, however, find that $\tilde{B}(x,\lambda)\sim\lambda^{2/(4+\lambda)}x^{2/(4+\lambda)}$ reaches, for large x, a maximum when $\lambda=4/\ln x$. What we claim is that the correct behavior in the limit $\lambda\to 0$ is the *envelope* of the curves $\tilde{B}(x,\lambda)$, which is obtained by assigning to λ the x-dependent value $4/\ln x$. Another way of understanding this procedure is to note that for a given (small) λ the system cannot know whether the correlation function of the disordered potential $f(x)$ is constant or actually decays until, say, $x^\lambda\simeq 2$ or $\lambda\sim 1/\ln x$. Computing all the numerical prefactors, we finally find that, in $d=3$,

$$\tilde{B}_L(x)=\begin{cases}\dfrac{3\Gamma(\frac{2}{3})}{7\pi^{2/3}}\left[\dfrac{x}{\xi}\right]^{1/3}\simeq 0.27\left[\dfrac{x}{\xi}\right]^{1/3} & \text{if } x\ll\xi\ ,\\[2ex] \dfrac{2^{1/2}}{3^{1/4}5}\left[\dfrac{\lambda x}{\xi}\right]^{1/2}\simeq 0.43\left[\dfrac{x}{\xi\ln x}\right]^{1/2} & \text{if } x\gg\xi\end{cases} \tag{54a}$$

and

$$\tilde{B}_T(x)=\begin{cases}\frac{4}{3}\tilde{B}_L(x) & \text{if } x\ll\xi\ ,\\ \frac{3}{2}\tilde{B}_L(x) & \text{if } x\gg\xi\ .\end{cases} \tag{54b}$$

Note that the logarithmic correction can be guessed from an Imry-Ma argument [see Eq. (7)].

2. Consequence for the correlation functions

The translational correlation function is usually defined as

$$g_K(\mathbf{x})=\overline{\langle e^{i\mathbf{K}\cdot[\mathbf{u}(\mathbf{x})-\mathbf{u}(0)]}\rangle}\ , \tag{55}$$

where $\mathbf{K}$ is an arbitrary vector. It is easy to show that within our Gaussian ansatz, $g_K(\mathbf{x})$ is given by

$$g_{\mathbf{K}}(\mathbf{x})=\exp\left[-\frac{K^2}{2}[\tilde{B}_L(x)\cos^2\theta+\tilde{B}_T(x)\sin^2\theta]\right]\ , \tag{56}$$

where θ is the angle between $\mathbf{K}$ and $\mathbf{x}$. Thus we predict that $g_{\mathbf{K}}(\mathbf{x})$ should be a stretched exponential, with a radial behavior depending on the (local) wandering exponent ν ($=\frac{1}{6}$ or $\frac{1}{4}$) as $\exp(-x^{2\nu})$. The angular anisotropy of the correlation function provides an independent measure of ν:

$$\ln g_{\mathbf{K}}(x,\theta=\pi/2)/\ln g_{\mathbf{K}}(x,\theta=0)=2\nu+1$$

(for large x). Another interesting quantity is the orientational correlation function, defined on a triangular lattice as

$$g_6(x)=\overline{\langle e^{6i[\varphi(\mathbf{x})-\varphi(0)]}\rangle}\ , \tag{57}$$

where $\varphi(\mathbf{x})$ is the angle made by a given lattice direction with a fixed axis: $\varphi \equiv \frac{1}{2}(\partial_x u_y - \partial_y u_x)$. We find that $g_6(x) = e^{-18B_{\rm or}(x)}$ with

$$B_{\rm or}(x) = \frac{T}{2}\int \frac{d^2q\, dq_z}{(2\pi)^3}[1-\cos(\mathbf{q}\cdot\mathbf{x})]q^2\widetilde{G}_T(q,q_z) \ . \tag{58}$$

From Eq. (23), with the proper values of ν, we find that $B_{\rm or}(x)$ does not grow for large x and thus that the long-range orientational order is not destroyed by disorder. This was also the case for the Larkin model in three dimensions; the term "hexatic vortex glass" was coined by Chudnovsky[13] to describe this situation where the translational order is destroyed while the orientational order is maintained. Note that we predict that $\partial B_{\rm or}(x)/\partial x$ behaves as $x^{2\nu-3}$ for large x.

B. Generalization to other dimensions: Discussion

1. Case of films

In the case of a bidimensional disordered solid (corresponding, e.g., to flux holes in thin superconducting films or to magnetic bubbles on a disordered substrate[3]), described by Eq. (9) with $C_4 \equiv 0$ and no z integral, similar calculations lead to the final result

$$\widetilde{B}_L(x) = \begin{cases} \dfrac{3\Gamma(\frac{2}{3})^2}{8\pi^{2/3}2^{1/3}}\left[\dfrac{x}{\xi}\right]^{2/3} \simeq 0.25\left[\dfrac{x}{\xi}\right]^{2/3} & \text{if } x \ll \xi \ , \\ \dfrac{1}{6}\left[\dfrac{\pi}{3^{1/2}}\right]^{1/2}\left[\dfrac{\lambda^{1/2}x}{\xi}\right] \simeq 0.32\dfrac{x}{\xi\sqrt{\ln(x)}} & \text{if } x \gg \xi \end{cases} \tag{59a}$$

and

$$\widetilde{B}_T(x) = \begin{cases} \frac{5}{3}\widetilde{B}_L(x) & \text{if } x \ll \xi \ , \\ 2\widetilde{B}_L(x) & \text{if } x \gg \xi \ , \end{cases} \tag{59b}$$

with, however, a different expression for the correlation length $\xi(d=2) \equiv C_T/\sqrt{W}$. Because we find $\nu < 1$, the orientational correlation function $B_{\rm or}(x)$ does not grow with x, and hence the orientational order is maintained even in $d=2$ (always neglecting dislocations). This is at variance with Chudnovsky's calculation, which leads to $g_6(x) \sim x^{-\alpha}$, where α is an exponent depending continuously upon the strength of the disorder in the form $\alpha \propto W/C_6^2$.

2. d-dimensional disordered solid

It is instructive to investigate the problem of a d-dimensional (isotropic) elastic solid in a random potential characterized by the Hamiltonian

$$H_d = \frac{1}{2}\int d^d\mathbf{x}\, C\left[\sum_{\alpha=1,d}\partial_\alpha u_\alpha\right]^2 + \sum_{\mathbf{x}} V(\mathbf{r}(\mathbf{x}),\mathbf{x}) \ , \tag{60}$$

with, e.g.,

$$\overline{V(\mathbf{r},\mathbf{x})V(\mathbf{r}',\mathbf{x}')} = W\delta^{(d)}(\mathbf{r}-\mathbf{r}')f(|\mathbf{x}-\mathbf{x}'|) \tag{61}$$

and $f(x) \simeq x^{-\lambda}$ for large x. The main difference with the vortex-glass case is the fact that $\mathbf{u}$ has d dimensions, instead of 2 in the case of "lines." The replica-symmetry-broken solution again distinguishes two regimes, separated by a crossover value v^*, which separates a random-manifold regime from a "correlated" regime (by which we mean that the fact that all "atoms" evolve in the same landscape is relevant). The growth of the fluctuations is then found to scale as

$$\widetilde{B}(x) \sim \left[\frac{x}{\xi}\right]^{2(4-d)/(4+d)} \ , \tag{62}$$

for $x \ll \xi \equiv (C^2/W)^\Upsilon$, with $\Upsilon = 1/(4-d)$ (random-manifold regime), and

$$\widetilde{B}(x) \sim \left[\frac{x}{\xi}\right]^{2(4-d)/(4+\lambda)} \ , \tag{63}$$

for $x \gg \xi$ (correlated regime). The formula $\nu = (4-d)/(4+d)$, which we obtain in the random-manifold regime, is a specialization of the general formula obtained in Ref. 6 for a d-dimensional manifold with n accessible dimensions in a short-ranged correlated potential:

$$\nu_{\rm MP} = \frac{4-d}{4+n} \ . \tag{64}$$

Here $n=d$, while in Sec. IV A, $d=3$, $n=2$. This formula for ν has the status of a Flory formula, since it can also be obtained by imposing that both terms in (54) behave similarly under rescaling.[6] The exact value of ν is not known. The approximation $\nu_{\rm MP}$ is clearly wrong for the "directed-polymer" case ($n=d=1$), where $\nu = \frac{2}{3}$.[23] Based on functional renormalization and heuristic arguments, Halpin-Healy[17] and Feigel'man *et al.*[15] have suggested the formula

$$\nu_{\rm HH} = \frac{4-d}{4+n/2} \ , \tag{65}$$

which gives the correct result for $n=d=1$. The exponents obtained from (65) for the vortex-glass problem are slightly higher than ours: $\nu = \frac{1}{5}$ instead of $\frac{1}{6}$ in $d=3$ and $\nu = \frac{2}{5}$ instead of $\frac{1}{3}$ in $d=2$. Let us mention that Fisher[24] has put forward general arguments which suggest that, in general, the random-manifold exponent obeys the bounds

$$\frac{4-d}{4+n} \le \nu \le \frac{4-d}{4} \ . \tag{66}$$

A calculation of Nattermann,[25] based on the same random-potential model [Eq. (3)] predicts $\nu=0$, i.e., a very slow, logarithmic, growth of $\widetilde{B}(x)$. We believe that the difference between our results and those obtained in Ref. 25 comes from an incorrect treatment of the potential energy in Ref. 25, which amounts to keep only subleading terms and discard the leading term. As above, in the limit $\lambda=0$, we find logarithmic corrections:

$$\tilde{B}(x) \sim \frac{\left[\frac{x}{\xi}\right]^{(4-d)/2}}{\sqrt{\ln x}} , \tag{67}$$

which is, again, the result obtained from a simple Imry-Ma argument [cf. Eq. (7b)]. In all the above cases, our theory predicts that the "energy" exponent ω is related to ν through $\omega = d - 2 + 2\nu$. While this relation is probably exact in the random-manifold regime, it is not obvious that it still holds in the correlated regime (see Ref. 21 for a related discussion).

C. Disorder with a nonzero correlation length and the replica-symmetric exponents

Let us turn now to the case where the correlation length $\Delta_{xy} \equiv \Delta a_0$ of the disordered potential is finite, but still much smaller than the lattice spacing a. Let us furthermore focus on the random-manifold regime ($x \ll \xi$ and $v > v^*$). The off-diagonal saddle-point equation (35) reads, in this limit and assuming replica-symmetry breaking,

$$\Sigma_{L,T}(q,q_z,v) = \frac{W}{2\pi T} \frac{1}{[B_{L,T}(0,v)+\Delta^2)]^2} . \tag{68}$$

Since $B_{L,T}(0,v)$ diverges for small v, one finds that there exists a new crossover value $\hat{v}$ such that $B_{L,T}(0,v) \gg \Delta^2$ for $v \ll \hat{v}$ and $B_{L,T}(0,v) \ll \Delta^2$ for $v \gg \hat{v}$. From Eq. (36) we find that

$$\hat{v} \propto \Delta^{-8} \frac{WT}{C_4 C_T^2} . \tag{69}$$

From Eq. (68) one thus finds that for $v \gg \hat{v}$, $\Sigma_{L,T}(q,q_z,v) = W/2\pi T \Delta^4$, while for $v \ll \hat{v}$ the previous results are recovered. This modifies the computation of the average displacements (Sec. III C 3): There exists now a *large-q, short-scale* regime in $\tilde{B}(x)$: If $C_T q^2 \gg [\sigma]_{\rm RM}(\hat{v})$ or $x \ll \xi_\Delta \equiv \Delta^6 \xi$, one finds

$$\tilde{B}(x) \propto \Delta^{-4} \frac{x}{\xi} = \Delta^2 \frac{x}{\xi_\Delta} , \tag{70}$$

which correctly crosses over to the random-manifold regime $[(x/\xi)^{1/3}]$ for $x = \hat{x}$. We thus recover the perturbative Larkin-Ovchinnikov (see also Ref. 26) exponent $\nu_{\rm LO} = (4-d)/2 = \frac{1}{2}$ if $\tilde{B}(x) \ll \Delta^2$, i.e., as long as the potential does not vary wildly. As discussed in Sec. II, this is reasonable, since in this case one may locally expand $V(x)$ and the perturbative results are expected to hold.

It is easy to show that the condition $\hat{v} \gg v^*$ coincides with $\Delta \ll 1$. In the other limit, the random-manifold regime entirely disappears, and along similar lines one establishes the behavior given in Sec. II [Eq. (8)].

Note that the appearance of the "perturbative" exponents at small scales (when the correlation length of the potential is nonzero) is probably very general and is valid beyond our variational approach. It would be interesting to observe this effect in the (1+1)-directed-polymer problem.

V. EXPERIMENTS: VORTEX LINES IN HT SUPERCONDUCTORS

It is now possible to observe the arrangements of the emerging flux lines over rather large regions of space: Bitter patterns containing 10 000 lines can be analyzed.[2] As the external field is increased, the density a_0^{-2} of the flux lattice increases $[(\sqrt{3}/2)Ha_0^2 = \phi_0]$. The elastic moduli also increase with the field since (see, e.g., Ref. 8) $C_{44} \sim C_{11} = H^2/4\pi$ and $C_{66} \simeq (H_{c2}/8\kappa^2 H)H^2/4\pi$, where κ is the Landau-Ginzburg parameter ($\kappa \simeq 200$ in those samples) and H_{c2} is the upper critical field. Taking the distance between planes to be $\Delta_z \simeq 1.5$ nm, we find that the following order of magnitudes for the scaled elastic moduli:

$$C_4 \simeq 2.1 \times 10^8 \text{ K}, \quad C_6 \simeq 6.7 \text{ K} ,$$

independently of H, and

$$C_1 \simeq 30H \text{ (K)} ,$$

with H expressed in gauss. (C_6 has been estimated here for $\kappa \simeq 200$ and $H_{c2} \simeq 10^5$ G. Let us recall that $H^2/4\pi = 6.5 \times 10^{-7}$ K/nm^3 for $H = 1$ G.)

The stiffening of the lattice with increasing field is accompanied by a gradual decrease of the density of dislocations. For the $H = 69$ G picture,[2] the distance between dislocations appears to be significantly larger than the picture size; and hence our dislocation-free description is justified. Our basic predictions in this regime are (i) that the translational correlation function decays as a stretched exponential and (ii) that the orientational order is maintained, both in three and two dimensions.

The truly asymptotic behaviors (i.e., beyond the distance between dislocations) cannot be addressed within the present method. Some conjectures have been put forward by Toner,[10] but these are based on a model in which the disorder is of the Larkin type. Another possibility would be to follow the point of view taken in Ref. 11, where the displacement u is decomposed into an *elastic* and a *dislocation-induced* part, and a phenomenological core energy for dislocations is introduced. Extension of the variational approach to this description would be interesting.

The data presented in Ref. 2, Fig. 2(a), concerns the angle-averaged $\langle g_{\mathbf{K}}(\mathbf{x}) \rangle_\theta$. The reported behavior is indeed consistent with a stretched exponential with $2\nu \sim 0.4 - 0.6$—a pure exponential fit does not go to one at the origin as it must. It should be noted that the effective correlation length $\xi_{\rm eff}$ defined by the distance for which $\langle g_{\mathbf{K}}(\mathbf{x}) \rangle_\theta = e^{-1}$, which is found to be ~ 10 (in lattice spacing units) is considerably smaller than the "bare" dimensional correlation length ξ of Eq. (48): In Ref. 2, $\mathbf{K}$ was taken to be a first reciprocal-lattice vector of magnitude $4\pi/3$, which means that (using our three-dimensional results [Eqs. (54) and (56)])

$$\xi = [64\pi^4 \Gamma(\tfrac{2}{3})^3/27] \xi_{\rm eff}(\theta=0) \approx 570 \xi_{\rm eff}(\theta=0) ,$$

so that, at least for these experiments, where the maximum length is $\simeq 50$ lattice spacings, we are presumably in the random-manifold $x \ll \xi$ regime, where $2\nu = \frac{1}{3}$.

This is comparable to the value which can be deduced from the experimental curve of Ref. 2. A few words of caution are, however, needed here.

(i) First, as we have emphasized in Sec. IV B 2, our value for ν is not expected to be exact, but it should be a good approximation to the true ν.

(ii) Second, it has been recently argued in Ref. 14 that *nonlocal elasticity* (i.e., the dependence of elastic moduli on q, in particular C_{44}) should be important in these samples, especially at "small" scales—x not much larger than 1. Such nonlocal corrections could be included if necessary. In the random-manifold regime, these corrections are easily obtained using the sum rule (A5) of Appendix A for $\tilde{G}$:

$$\tilde{G}_{L,T}(q,q_z)=\frac{1}{C_{L,T}q^2+C_4(q,q_z)q_z^2}\left[1+\int_0^1\frac{dw}{w^2}\,\frac{[\sigma](w)}{C_{L,T}q^2+C_4(q,q_z)q_z^2+[\sigma](w)}\right], \tag{71}$$

with (for example, see Ref. 14)

$$C_4(q,q_z)=C_4(0,0)\frac{1}{1+\lambda^2(q^2+q_z^2)}$$

and $[\sigma]$ given in Eq. (38).

(iii) Last, it has been argued by Huse[27] that emerging flux lines should rather be seen as an *effectively two-dimensional* system, i.e., that the configuration adopted by these flux lines is entirely determined by surface disorder and interactions between the surface magnetic charges. In other words, a Bitter pattern would not be equivalent to planar "cuts" of the flux lattice. This important question could be answered through a detailed fit of the data to our theoretical predictions. The exponent 2ν, in particular, is a good indicator of the dimensionality; it would equal $\frac{2}{3}$ in the Huse scenario, which seems to be appreciably larger than the experimental value. It should also be kept in mind that the experimental samples have a finite thickness L_z. A rough estimate of the crossover thickness between 2D and 3D behavior can be read from the q,q_z dependence of our 3D propagators in $1/(C_4q_z^2+C_6q^2)$: If one considers the system on some scale L_{xy} in the layers, the 2D-3D crossover should be at $L_z\sim L_{xy}\sqrt{C_4/C_6}$. As L_{xy} is measured in units of the vortex-lattice spacing and L_z is measured in units of the interlayer spacing, this gives, for $L_{xy}=10$, a typical estimate of L_z of 20 μm for the above values of C_4,C_6. It may thus be that the experiments of Ref. 2 are in the crossover regime.

Our detailed theory for the correlation functions also allows us to address the following question. What may be the pinning centers responsible for the destruction of translational order in these samples? The experimental data suggest (see above) $\xi\equiv C_4^{1/2}C_6^{3/2}/W\sim 5700$. As we have seen above, a reasonable estimate for $C_4^{1/2}C_6^{3/2}$ is 8×10^4 K^2, while the definition of W gives $W=(2\pi)^{3/2}(\Delta_{xy}/a_0)^2U_p^2$, where a_0 is the intervortex spacing, Δ_{xy} is the correlation length of the disorder in the xy plane, and U_p is the typical pinnning energy "felt" by one vortex. We thus get the following estimate for the product $U_p\Delta_{xy}$:

$$U_p\Delta_{xy}\sim 500\ \text{K nm}\ . \tag{72}$$

This result seems to be reasonable for the pinning by objects of atomic size such as, for instance, oxygen vacancies, where one could expect a value of Δ_{xy} around 5–10 nm. If one considers a more specific model for pinning by oxygen vacancies such as the one in Ref. 28, one should write

$$U_p^2\sim n_0\Delta_z\pi\Delta_{xy}^2\left[\frac{H_c^2}{4\pi}\,\frac{\pi D^2\xi_0}{4}\right]^2, \tag{73}$$

where n_0 is the concentration of oxygen vacancies per unit volume, $\xi_0\sim 3$ nm is the superconducting coherence length, Δ_z the interlayer spacing, $D\sim 0.4$ nm the "diameter" of the vacancy, and $H_c\sim 5000$ G the critical field (i.e., the condensation energy). Then one can account for the above figure for $U_p\Delta_{xy}$ by choosing the reasonable values $n_0\sim 0.3$ nm^{-2} and $\Delta_{xy}\sim 2\xi_0-3\xi_0$. Using the Larkin-Ovchinnikov model, Chudnovsky[13] has also argued that pinning by oxygen vacancies was compatible with the observed order of magnitude for ξ. However, our analysis indicates that the experiments probe the random-manifold regime rather than the very-short-distance regime where Larkin's model holds. In the former regime, $\xi_0=\Delta_{xy}\ll a_0$, and hence $\xi\equiv C_4^{1/2}C_6^{3/2}/W$ should be a *decreasing* function of the field: $\xi\propto H^{-1}$ ($H^{-1/2}$ for films). Only in the Larkin-Ovchinnikov regime given by Eq. (6a) does the effective correlation length ξ_Δ/Δ^2 grow linearly with the field, as found by Chudnovsky.[13] Unfortunately, the experiments for small fields correspond to lattices which are plagued with dislocations, for which our theory ceases to apply. Larger fields are needed to check our prediction, with the problem that the lattice spacing will soon become rather small and Bitter decoration harder to keep neat. It would, however, be crucial to confirm the hypothesis of a *microscopic* (i.e., $\Delta\ll 1$) disorder.

It would also be of great interest to check our predictions on the two-dimensional magnetic-bubble system studied in Ref. 3, where deviations from the Larkin-Ovchinnikov exponents should be even more appreciable.

VI. PERSPECTIVES

The detailed microscopic analysis which we have performed in this paper only studies the static equilibrium properties of the vortex lattice. It would be, of course, extremely interesting to extend this type of method to dynamical problems in order to address quantitatively the major issues of the value of the critical current or the voltage-current relation. Unfortunately, such an approach has not been developed yet (see, however, an attempt in Ref. 33).

Therefore we shall not expand on the dynamics here, but just present a few remarks. As suggested in Refs. 4, 12, 15, 19, and 29, purely static (equilibrium) calculations may be used to guess some aspects of the dynamic problem. The order of magnitude of the zero-temperature critical current $J_c(T=0)$, for example, is obtained by noting that the random forces acting on each vortex essentially add in a random way until the scale ξ_Δ: Above this scale the vortices can adapt to the local potential and the pinning forces start contributing coherently. $J_c(T=0)$ is thus the result of an "interrupted averaging" at scale ξ_Δ of the local forces $F_{\max} \simeq U_p/\Delta$. Introducing back the physical units of length, the correlation volume becomes $\xi_\Delta^2 a_0^2 \xi_\Delta(\sqrt{C_4/C_6})\Delta_z$, and thus the total pinning force density is

$$\frac{F_p}{V} = \frac{U_p}{\Delta_{xy}} \xi_\Delta^{3/2} \left[\frac{C_4}{C_6} \right]^{-1/4} \frac{1}{a_0^2 \Delta_z} \underset{\text{def}}{\equiv} H J_c(T=0) \ . \quad (74)$$

This expression holds while $\xi_\Delta \gg 1$; otherwise, the vortices adapt individually to the pinning forces and one should set $\xi_\Delta \equiv 1$. The rough estimate quoted above based on pinning by oxygen vacancies leads to a reasonable $J_c(T=0) \sim 3\times 10^7$ A/m^2.

Our approach also justifies a frequent hypothesis of self-similarity of the energy landscape: Two metastable configurations which differ on scale $x \simeq q^{-1}$ have a (free) energy difference of order $\Delta F \propto x^\omega$. An estimate based on thermal activation over energy barriers[15,25,30] then leads to a current which decays as a power of $1/[T\ln(t)]$ and to a nonlinear voltage-current relation

$$\mathcal{V} \propto \exp\left[-\left[\frac{J_0}{J} \right]^\mu \right] , \quad (75)$$

with $\mu = \omega/(d+\nu-\omega)$. The linear conductivity $\partial J/\partial \mathcal{V}|_{J=0}$ is thus zero.[29] The value of μ depends on the relative position of x_J, ξ, ξ_Δ, and 1. We find, in particular,

$$\mu = \begin{cases} \frac{8}{11} & \text{if } 1, \xi_\Delta \ll x \ll \xi \ , \\ \frac{6}{7} & \text{if } x \gg \xi \ , \end{cases} \quad (76a)$$

for three-dimensional samples, and

$$\mu = \begin{cases} \frac{2}{5} & \text{if } 1, \xi_\Delta \ll x \ll \xi \ , \\ \frac{2}{3} & \text{if } x \gg \xi \ , \end{cases} \quad (76b)$$

for films. Note that the experimental results of Koch *et al.*[31] on epitaxially grown films of Y-Ba-Cu-O suggest $\mu = 0.4 \pm 0.2$, in rather good agreement with our result for $d=2$ in the random-manifold regime.

As a conclusion, let us summarize our central result: We have shown that a variational treatment of the disordered vortex-lattice problem, based on some Gaussian ansatz, requires one to break the replica symmetry in order to get sensible results. The solution we obtain can be used to predict various correlation functions. We have shown that three regions of space must be distinguished: a short-scale region, where the (replica symmetric) Larkin-Ovchinnikov results are recovered, an intermediate-scale region for which the random-manifold exponents hold, and a large-scale region, where the typical displacement exceeds one lattice spacing and where new exponents are found. (Another regime, where dislocations play a role, has not been considered here.) We have compared our results with experimental data and found promising agreement. More precise fits, on large-scale pictures recently obtained by the Bell group[32] and on the magnetic-bubble problem, would certainly be welcome: These experimental systems are remarkable in that they permit a direct observation of microscopic configurations. They provide an excellent testing ground for theoretical approaches to the physics of disordered systems which have been developed during the last two decades.

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APPENDIX A: INVERSION OF HIERARCHICAL MATRICES AND USEFUL SUM RULES

We give here, for convenience and in the notations used throughout the paper, the rules for inverting Parisi's matrices in the $n=0$ limit (these are derived for instance, in Ref. 6). Defining the "connected" part G^c of G as (indices L or T are omitted)

$$G^c(q,q_z) = G^{aa}(q,q_z) + \sum_{a\neq b} G^{ab}(q,q_z) \ , \quad (A1)$$

one has

$$G^c(q,q_z) = \frac{1}{(G^{-1})^c(q,q_z)} \ . \quad (A2)$$

We denote by $-\Sigma_{ab}$ the off-diagonal ($a\neq b$) part of the inverse of G. For $n\to 0$, the two functions $G(q,q_z,v)$ and $\Sigma(q,q_z,v)$ describing the off-diagonal elements of G and its inverse are related through

$$G(q,q_z,v) = G_c(q,q_z) \left[\frac{[\Sigma](q,q_z,v)}{v\,[G_c(q,q_z)^{-1} + [\Sigma](q,q_z,v)]} + \int_0^v \frac{dw}{w^2} \frac{[\Sigma](q,q_z,w)}{G_c(q,q_z)^{-1} + [\Sigma](q,q_z,w)} + G_c(q,q_z)\Sigma(q,q_z,0) \right] , \quad (A3)$$

where the bracket transform $[f]$ of a function $f(v)$ is defined as

$$[f](v)=vf(v)-\int_0^v dw\, f(w)\ . \tag{A4}$$

Two sum rules are often useful:

$$\tilde{G}(q,q_z)=G_c(q,q_z)\left[1+\int_0^1 \frac{dw}{w^2}\frac{[\Sigma](q,q_z,w)}{G_c(q,q_z)^{-1}+[\Sigma](q,q_z,w)}+G_c(q,q_z)\Sigma(q,q_z,0)\right] \tag{A5}$$

and, for $v>0$,

$$\tilde{G}(q,q_z)-G(q,q_z,v)=\left[\frac{1}{v[G_c(q,q_z)^{-1}+[\Sigma](q,q_z,v)]}-\int_v^1 \frac{dw}{w^2}\frac{1}{G_c(q,q_z)^{-1}+[\Sigma](q,q_z,w)}\right]\ . \tag{A6}$$

APPENDIX B: DEFINITION AND CALCULATIONS OF SOME INTEGRALS

Here we define some relevant integrals used in the text and give their numerical values. From (30) and (32), one has

$$\hat{\sigma}(0)\equiv\hat{\sigma}_{L,T}(0)=\sigma_{L,T}(0)\frac{(2+\lambda)\nu}{\omega+(2+\lambda)\nu}=\sigma_{L,T}(0)\left[1-\frac{\omega}{2}\right] \tag{B1}$$

and

$$\sigma_L(0)\equiv\frac{2}{\sqrt{3}}\int\frac{d^2z}{2\pi}e^{-z^2/2}[(1-z^2)\cos^2\phi+\sin^2\phi]z^{-\lambda}=\frac{2}{\sqrt{3}}\frac{\lambda}{2}\Gamma\left[1-\frac{\lambda}{2}\right]2^{-\lambda/2}\ . \tag{B2}$$

From (27), one has

$$b_0\equiv b_{L,T}(0)=2N\int\frac{d^2k}{(2\pi)^3}\frac{1}{k^{2+2\nu}}\left[h_\omega(\infty)-h_\omega\left[\frac{N}{k^\omega}\right]\right]\sin^2(\phi)=\frac{N^{1-2\nu/\omega}}{4\pi(\omega-1)}\ . \tag{B3}$$

From (45), one has

$$K(\omega')=\int_0^\infty\frac{dk}{2\pi}\left[\frac{1}{(1+k^{-2})^{1/2}}-\int_1^\infty\frac{du}{u^2}\frac{1}{(1+k^{-2}u^{2/\omega'})^{1/2}}\right]=\frac{1}{2\pi(\omega'-1)}=\frac{3}{2\pi}\ . \tag{B4}$$

From (50), one has

$$N_s(\nu)=\int_0^\infty\frac{dq}{(2\pi)^3}q^{-1-2\nu}\int_0^{2\pi}d\phi[1-\cos(q\cos\phi)]\sin^2\phi=\frac{1}{2^{4+2\nu}\pi^2}\frac{\Gamma(1-\nu)}{\nu^2(1+\nu)\Gamma(\nu)} \tag{B5}$$

and

$$N_c(\nu)=\int_0^\infty\frac{dq}{(2\pi)^3}q^{-1-2\nu}\int_0^{2\pi}d\phi[1-\cos(q\cos\phi)]\cos^2\phi=(2\nu+1)N_s(\nu)\ . \tag{B6}$$

Finally, from (24), one has

$$h_\omega(\omega)=\pi\int_0^\infty\frac{dv}{v^2}\{1-(1+v^{2/\omega})^{-1/2}\}=\sqrt{\pi}\Gamma\left[1-\frac{\omega}{2}\right]\Gamma\left[\frac{1}{2}+\frac{\omega}{2}\right]\ . \tag{B7}$$

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