Variational Theory for Disordered Vortex Lattices

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We derive a variational replica-symmetry-breaking theory for the effect of random impurities on two- and three-dimensional vortex lattices. We find that the translational correlation functions decay as stretched exponentials with exponents which seem to be in good agreement with experiments. We predict, in the absence of dislocations, long-range orientational order in three *and two* dimensions.

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Recent Bitter decoration [1] and transport experiments on high-temperature superconductors have reawakened interest in the classic problem of the effect of impurities on the Abrikosov lattice of vortex lines in type-II superconductors [2,3]. The two-dimensional (thin film) version of this system is closely related to lattices of magnetic bubbles on disordered substrates, for which experiments have also been reported very recently [4]. This problem was first studied theoretically by Larkin [5], whose theory has since been elaborated by numerous authors [6–10]. Unfortunately, the exactly soluble model introduced by Larkin has unphysical features, which we shall describe below in more detail. In this Letter, we introduce a more realistic model and study it using a variational replica field theory approach which has recently been applied to the related problem of a directed manifold in a random potential [11]. Our predictions are quite different from those derived using the Larkin model, and are in good agreement with experiments.

The Larkin Hamiltonian is $H = H_{\text{elastic}} + H_{\text{pin}}$. H_{elastic} is a continuum elastic description of the triangular Abrikosov lattice:

$$H_{\text{elastic}} = \frac{1}{2}\int d^2x\, dz \left[(C_{11} - C_{66}) \left(\sum_\alpha \partial_\alpha u_\alpha \right)^2 + C_{66} \sum_{\alpha,\beta} (\partial_\alpha u_\beta)^2 + C_{44} \sum_\alpha (\partial_z u_\alpha)^2 \right], \tag{1}$$

where α and β are indices denoting the x and y directions, $\mathbf{x}$ and z are "internal" coordinates labeling the vortex lines in the x-y plane and z directions, $\mathbf{u}(\mathbf{x},z)$ is the fluctuation in the x-y plane of the vortex line from its equilibrium position $\mathbf{x}$, and C_{11}, C_{66}, and C_{44} are the bulk, shear, and tilt moduli [12]. (We consider the three-dimensional problem here and below unless referring explicitly to the two-dimensional case.) This elastic Hamiltonian assumes an underlying crystal lattice and therefore does not describe a possible entangled phase [3]. It also implicitly excludes dislocations, but that should not be too serious a problem for comparison with experiments in the high-vortex-density regime where dislocations seem to be very rare [1,4]. The effects of disorder are described with a pinning Hamiltonian $H_{\text{pin}} = \int d^2x\, dz\, \mathbf{u}(\mathbf{x},z) \cdot \mathbf{f}(\mathbf{x},z)$ in which each vortex is subjected to an *independent random force*. In Larkin's model, two different vortices which wander to the same point in space at different times feel *different* pinning forces when they are at that point, because the random force is assigned according to the label of the vortex, and does not depend on its position in space. A related unrealistic feature of this Hamiltonian is that there are no metastable configurations of the vortex lines.

In our model, we use the same elastic Hamiltonian, but take a pinning potential that depends on the position in space of the vortices (although for technical reasons, it is convenient to consider a general model where the pinning potential also depends on the internal label $\mathbf{x}$). We take care to include the discrete nature of the vortices, which can have an important effect on the results (although one can still safely use a continuum elastic energy). Thus we write $H_{\text{pin}} = \int dz \sum_{\mathbf{x}} V(\mathbf{r}(\mathbf{x},z),z,\mathbf{x})$, where $\mathbf{r}(\mathbf{x},z) = \mathbf{x} + \mathbf{u}(\mathbf{x},z)$ is the position of a vortex line and V is a Gaussian random pinning potential with zero mean. We assume

$$\overline{V(\mathbf{r},z,\mathbf{x})V(\mathbf{r}',z',\mathbf{x}')} = \frac{U_p^2}{\Delta_z^2} \exp\left[-\frac{(\mathbf{r}-\mathbf{r}')^2}{2\Delta_{xy}^2} - \frac{(z-z')^2}{2\Delta_z^2} \right] \times f\left(\frac{|\mathbf{x}-\mathbf{x}'|}{a_0} \right), \tag{2}$$

where the overline denotes a disorder average and a_0 is the lattice spacing in the x-y plane. We introduce a regularization function $f(x)$ which behaves as $x^{-\epsilon}$ for large x, and use a very small, but positive ϵ—the physical limit we are interested in is $\epsilon \to 0$. It is interesting to note that if $f(x)$ were instead a delta function, our model would become identical to a $d=3$ dimensional "directed" manifold with $n=2$ transverse components in a random potential. The physical difference between our model and that one is that, in our model, all the vortex lines see the same pinning potential, while for the "random manifold" problem, each vortex line sees an *independent* pinning potential.

To make analytical progress in computing the free energy F, we first average over the disorder using the replica method. We find an effective replica Hamiltonian:

$$H_n=\frac{1}{2}\sum_a\int d^2x\,dz\left[(C_1-C_6)\left[\sum_\alpha \partial_\alpha u^a_\alpha\right]^2+C_6\sum_{\alpha\beta}(\partial_\alpha u^a_\beta)^2+C_4\sum_\alpha(\partial_z u^a_\alpha)^2\right]$$
$$-\frac{W}{2T}\sum_{ab}\sum_{\mathbf{x}\mathbf{x}'}\int dz\,\delta^{(2)}(\mathbf{x}+\mathbf{u}^a(\mathbf{x},z)-\mathbf{x}'-\mathbf{u}^b(\mathbf{x}',z))f(|\mathbf{x}-\mathbf{x}'|)\,. \tag{3}$$

In this formula, all distances are written in units of a_0 in the x-y plane, and in units of Δ_z in the z direction. We have taken the limit $\Delta_z,\Delta_{xy}\ll a_0$ and introduced $C_1\equiv C_{11}a_0^2\Delta_z$, $C_6\equiv C_{66}a_0^2\Delta_z$, and $C_4\equiv C_{44}a_0^4\Delta_z^{-1}$ which have dimensions of an energy, as well as $W\equiv(2\pi)^{3/2}U_p^2\Delta_{xy}^2a_0^{-2}$ which has the dimension of an energy squared. T is the temperature and a and b are replica indices.

Our method is to find the best quadratic approximation of the Hamiltonian. We take as a trial Hamiltonian (in Fourier space)

$$H_0=\frac{1}{2}\int\frac{d^2q\,dq_z}{(2\pi)^3}\sum_{ab}\sum_{\alpha\beta}u^a_\alpha(\mathbf{q},q_z)(G^{-1})^{\alpha\beta}_{ab}(\mathbf{q},q_z)u^b_\beta(-\mathbf{q},-q_z) \tag{4}$$

and use the convexity inequality $\tilde F(G)\equiv F_0+\langle H-H_0\rangle_0\ge F$ to find the best G, minimizing $\tilde F(G)$. We find the saddle-point equations

$$[G_L^{-1}]_{aa}(q,q_z)=C_1q^2+C_4q_z^2+\frac{W}{2\pi T}\sum_{\mathbf{x}}\frac{e^{-x^2/2B_L^{aa}(x)}f(x)}{[B_L^{aa}(x)B_T^{aa}(x)]^{1/2}}[1-\cos(\mathbf{q}\cdot\mathbf{x})]\left[\frac{\sin^2\phi}{B_T^{aa}(x)}+\frac{\cos^2\phi}{B_L^{aa}(x)}\left(1-\frac{x^2}{B_L^{aa}(x)}\right)\right]$$
$$+\frac{W}{2\pi T}\sum_{b(\neq a)}\sum_{\mathbf{x}}\frac{e^{-x^2/2B_L^{ab}(x)}f(x)}{[B_L^{ab}(x)B_T^{ab}(x)]^{1/2}}\left[\frac{\sin^2\phi}{B_T^{ab}(x)}+\frac{\cos^2\phi}{B_L^{ab}(x)}\left(1-\frac{x^2}{B_L^{ab}(x)}\right)\right] \tag{5a}$$

and

$$[G_L^{-1}]_{a\neq b}(q,q_z)=-\frac{W}{2\pi T}\sum_{\mathbf{x}}\frac{e^{-x^2/2B_L^{ab}(x)}f(x)}{[B_L^{ab}(x)B_T^{ab}(x)]^{1/2}}\cos(\mathbf{q}\cdot\mathbf{x})\left[\frac{\sin^2\phi}{B_T^{ab}(x)}+\frac{\cos^2\phi}{B_L^{ab}(x)}\left(1-\frac{x^2}{B_L^{ab}(x)}\right)\right], \tag{5b}$$

where G_L is the longitudinal component of G [13], and

$$B_L^{ab}(x)=T\int\frac{d^2q\,dq_z}{(2\pi)^3}\{[G_L^{aa}(q,q_z)+G_L^{bb}(q,q_z)-2G_L^{ab}(q,q_z)\cos(\mathbf{q}\cdot\mathbf{x})]\cos^2\phi$$
$$+[G_T^{aa}(q,q_z)+G_T^{bb}(q,q_z)-2G_T^{ab}(q,q_z)\cos(\mathbf{q}\cdot\mathbf{x})]\sin^2\phi\}\,. \tag{6}$$

(ϕ is the angle between $\mathbf{q}$ and $\mathbf{x}$.) The corresponding formulas for the transverse components G_T and B_T are obtained by replacing C_1 with C_6 and by inverting the roles of $\cos^2\phi$ and $\sin^2\phi$. The disorder average of the fluctuations is determined from the diagonal components of B; for example,

$$\tilde B_L(x)\equiv B_L^{aa}(x)=\overline{\langle\{[\mathbf{u}(\mathbf{x},z)-\mathbf{u}(0,z)]\cdot\mathbf{x}/x\}^2\rangle}\,. \tag{7}$$

The "wandering" exponent ν is defined by $\tilde B_{L,T}(x)\sim x^{2\nu}$ for large x.

The simplest solution to those equations is the "replica symmetric" one, with $\tilde G\equiv G^{aa}$ and $G\equiv G^{a\neq b}$. The replica symmetric ansatz gives scaling results that are qualitatively similar to those derived from the Larkin model: For example, $\tilde B_L(x)$ grows for large x as x/ξ_{RS} in three dimensions, and as $(x/\xi_{\mathrm{RS}})^2$ in two dimensions. This replica symmetric solution is, however, not satisfactory; for instance, in three dimensions, this is signaled by the fact that ξ_{RS} goes to zero as $(C_L/C_4)^{1/2}T^2/W$ at low temperatures, which is clearly unphysical. Furthermore, there are many metastable configurations of the vortices within our model. The nature of the metastable configurations is physically very similar to those in the random manifold problem where it has been shown [11] that the replica symmetric solution is unstable and that replica symmetry breaking is needed to more correctly account for the effects of those metastable configurations. We thus look for a Parisi replica-symmetry-broken solution, following Ref. [11]. The replica indices $a\neq b$ become, in the limit $n=0$, a continuous variable $0\le v\le 1$ [14] so that the G's and B's become functions of (q,q_z) (respectively, x) *and* v. For reasons of space, we can only give here the main results of our replica-symmetry-breaking solution; a full derivation will be given elsewhere [15].

We have found a solution (for small q and v) of the form

$$\int_{-\infty}^{+\infty} dq_z\, G_{L,T}(q,q_z,v) = \frac{1}{C_4 q^{2+2\nu}} g_{L,T}\left(\frac{v}{q^{\omega}}\right) \qquad (8)$$

and

$$B_{L,T}(x,v) = \frac{T}{C_4} v^{-2\nu/\omega} b_{L,T}(x v^{1/\omega}) \qquad (9)$$

(but with two different sets of exponents ν and ω and scaling functions g and b for v much greater or much less than a certain v_* discussed below). ω turns out to be the "energy" exponent, related to ν by $\omega = 2\nu+1$ ($\omega = 2\nu$ in two dimensions) [16]. Assuming that $C_{66} \ll C_{11}$, we find that the renormalized shear modulus C_T is still much smaller than the renormalized bulk modulus C_L.

Because of the divergence of B for small v, we must deal separately with two regimes in v. For $v \ll v_* \simeq \epsilon WT/32\pi^2 C_4 C_T^2$ [derived using $B_L(1,v_*)\equiv 1$], we can approximate the sums over $\mathbf{x}$ in Eqs. (5) as integrals. In this regime, we find $\nu = 1/(4+\epsilon)$, $\omega = (6+\epsilon)/(4+\epsilon)$, and $g_{L,T}(t) = (C_4/C_{L,T})^{1/2} \gamma_{L,T}^{\omega/2} \tilde{g}(t\gamma_{L,T}^{\omega/2})$, where $\gamma_{L,T} \simeq \epsilon W/8TC_{L,T} v_*^{1/3}$ and

$$\tilde{g}(t) = \pi \left\{ t^{-1}[1-(1+t^{2/\omega})^{-1/2}] + \int_0^t dv\, v^{-2}[1-(1+v^{2/\omega})^{-1/2}] \right\} .$$

For $1 \gg v \gg v_*$, we can, because of the exponential damping factor, approximate the sums in Eq. (5) by keeping only the $\mathbf{x}=\mathbf{0}$ term. We find that in an intermediate regime of q ($1 \gg q \gg \xi^{-1} \equiv W/C_4^{1/2} C_T^{3/2}$), the scaling form given above still holds, albeit with different exponents: $\nu = \frac{1}{6}$, $\omega = \frac{4}{3}$ (and with different $\gamma_{L,T}$). In this regime, the saddle-point equations are essentially identical to the ones analyzed in [11] for the random manifold problem. The final results for the interesting physical quantities $\tilde{B}_L(x)$ and $\tilde{B}_T(x)$ (which measure the growth of longitudinal and transverse fluctuations with distance) are

$$\tilde{B}_L(x) = \frac{3}{4}\tilde{B}_T(x) = \frac{3\Gamma(\frac{2}{3})}{7\pi^{2/3}} \left(\frac{x}{\xi}\right)^{1/3} \simeq 0.2705 \left(\frac{x}{\xi}\right)^{1/3} \qquad (10)$$

for $1 \ll x \ll \xi$, and

$$\tilde{B}_L(x) = \frac{2}{3}\tilde{B}_T(x) = \frac{2^{1/2}}{3^{1/4}5} \left(\frac{\epsilon x}{\xi}\right)^{1/2} \simeq 0.43 \left(\frac{x}{\xi \ln x}\right)^{1/2} \qquad (11)$$

for $x \gg \xi$. (To extract the scaling results in the limits $\epsilon \to 0$, $x \to \infty$, one should set $\epsilon \simeq 4/\ln x$ [15].) Note that the ratio $\tilde{B}_T/\tilde{B}_L$ is always precisely equal to $2\nu+1$ (assuming $C_T \ll C_L$). The correction due to disorder to the renormalized shear and bulk moduli C_T and C_L is positive and small (for large ξ): $\Delta C_6/C_6 \sim (1/\xi)^2$, $\Delta C_1/C_1 \sim (C_6/C_1)(1/\xi)^2$.

Our solution of the two-dimensional model is very similar; the final result is that

$$\tilde{B}_L(x) = \frac{3}{5}\tilde{B}_T(x) = \frac{3\Gamma(\frac{2}{3})^2}{8\pi^{2/3}2^{1/3}} \left(\frac{x}{\xi}\right)^{2/3} \simeq 0.2544 \left(\frac{x}{\xi}\right)^{2/3} \qquad (12)$$

for $1 \ll x \ll \xi \equiv C_T/\sqrt{W}$, and

$$\tilde{B}_L(x) = \frac{1}{2}\tilde{B}_T(x) = \frac{1}{6}\frac{\pi^{1/2}}{3^{1/4}} \left(\frac{\epsilon^{1/2} x}{\xi}\right) \simeq 0.317 \left(\frac{x}{\xi\sqrt{\ln x}}\right) \qquad (13)$$

for $x \gg \xi$. (In this case, we set $\epsilon \simeq 2/\ln x$.)

In Refs. [17,18], the problem we are considering was assumed to be equivalent to the random manifold problem discussed previously [19], where all the vortices feel an independent random potential. Our results show that this equivalence is valid only for short length scales $x \ll \xi$. At larger length scales, the $v \ll v_*$ region dominates, and changes the universality class (experimentally, the crossover might be difficult to identify, as the exponents in the two regimes are rather close). The crossover at $x = \xi$ can be easily understood: ξ is the distance at which vortex lines are typically displaced from their unperturbed position by an amount comparable to the lattice spacing a_0. Until that length scale, two neighboring vortices can indeed be considered to be seeing nearly independent potentials. Note that v_* has an important physical significance—T/v_* is the typical value of the energy fluctuations at scale ξ, and is thus related to the value of the critical current [15].

We now briefly discuss the relation of our theory to experimental results. We find that the translational correlation function $g_{\mathbf{K}}(\mathbf{x}) \equiv \langle \rho_{\mathbf{K}}(\mathbf{x}) \rho_{\mathbf{K}}^*(\mathbf{0}) \rangle$ (where $\rho_{\mathbf{K}}(\mathbf{x}) \equiv \exp[i\mathbf{K}\cdot\mathbf{u}(\mathbf{x},z)]$ and $\mathbf{K}$ is an arbitrary vector) is given by

$$g_{\mathbf{K}}(\mathbf{x}) = \exp\{-\tfrac{1}{2}K^2[\tilde{B}_L(x)\cos^2\theta + \tilde{B}_T(x)\sin^2\theta]\} , \qquad (14)$$

where θ is the angle between $\mathbf{K}$ and $\mathbf{x}$. Thus we predict that $g_{\mathbf{K}}(\mathbf{x})$ should be a stretched exponential, with a radial behavior depending on the wandering exponent ν as $\exp(-x^{2\nu})$. The angular anisotropy of the correlation function provides an independent measure of ν: $\ln g_{\mathbf{K}}(x, \theta=\pi/2)/\ln g_{\mathbf{K}}(x,\theta=0) = 2\nu+1$ (for large x).

In three dimensions, we find a "hexatic vortex glass" as described by Chudnovsky [7], with long-range orientational order, but no long-range translational order. Chudnovsky's calculation of $g_{\mathbf{K}}(\mathbf{x})$ based on the Larkin model [7] gives a result similar to ours, but with a quantitatively quite different value of $\nu = \frac{1}{2}$ (and $\nu = 1$ in two dimensions.) In fact, the data presented for the angle-averaged $g_{\mathbf{K}}(\mathbf{x})$ for the three-dimensional 69-G experiments in Ref. [1] are clearly more consistent with a stretched exponential with $\nu \sim 0.2$–0.3—a pure exponential fit does not go to 1 at the origin as it must. It should be noted that the effective correlation length $\xi_{\rm eff}$ defined by the distance for which $g_{\mathbf{K}}(\mathbf{x})=e^{-1}$ may be considerably smaller than our "bare" dimensional correlation length ξ. For example, in Ref. [1], $\mathbf{K}$ was taken to be a first reciprocal-lattice vector of magnitude $4\pi/\sqrt{3}$, which means that (using our 3D random manifold results) $\xi=[512\pi^4\Gamma(\frac{2}{3})^3/343]\xi_{\rm eff}(\theta=0)\approx 360\xi_{\rm eff}(\theta=0)$, so that at least for these experiments, we are presumably in the $x\ll\xi$ regime. In two dimensions, we find that orientational order is not destroyed, in contrast with the results of Ref. [7]. We also predict a *slower* than exponential decay for $g_{\mathbf{K}}(\mathbf{x})$. It would be interesting to compare these results with the experiments on magnetic bubbles [4]. The $x\gg\xi$ regime should be more easily accessible in two dimensions.

A more precise experimental and theoretical determination of the exponents and correlation functions would yield crucial information on the nature of pinning in these systems. On the theoretical side, it might be useful to include the effects of nonlocal elastic constants [10] and compute the full correlation functions beyond the asymptotic scaling regimes.

The experimental systems that we have discussed are remarkable in that they permit a direct observation of microscopic configurations. They provide an excellent testing ground for some of the theoretical approaches to the physics of disordered systems which have been developed during the last two decades.

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