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The Euclidean matching problem

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Résumé. — Nous étudions le problème du couplage (« matching ») en dimension finie. Les corrélations euclidiennes entre les distances peuvent être prises en compte de manière systématique. Par rapport au cas des distances aléatoires indépendantes que nous avions étudiées précédemment, les corrélations triangulaires euclidiennes engendrent des corrections qui s'annulent dans la limite où la dimension de l'espace tend vers l'infini, et restent relativement petites à toute dimension.

Abstract. — We study the matching problem in finite dimensions. The Euclidean correlations of the distances can be taken into account in a systematic way. With respect to the case of independent random distances which we have studied before, the adjonction of Euclidean triangular correlations gives rise to corrections which vanish when the dimension of space goes to infinity, and remain relatively small in any dimensions.

The matching problem is a rather simple system which has similarities with a spin glass with finite range interactions [1]. The problem is simply stated : given $2N$ points, and a matrix of distances between them, find the perfect matching between the points (a set of unoriented links such that each point belongs to one and only one link), of shortest length. In more mathematical terms, if ℓ_{ij} is the distance between points i and j (we keep to the symmetric case $\ell_{ij} = \ell_{ji}$), one looks for a set of link occupation numbers $n_{ij} = 0$ or 1, such that :

$$\forall i : \sum_j n_{ij} = 1 ,$$

minimize

$$E = \sum_{i<j} n_{ij} \ell_{ij} . \tag{1}$$

Given an instance of the problem, i.e. the matrix ℓ, finding a solution of equation (1) is numerically a polynomial problem, which can be solved by rather efficient algorithms [2]. Analytically we would like to compute the most likely value of E for large N. The case of the *random link* model where the distances are independent random numbers, identically distributed, has been studied in great detail. The aim of this note is to extend this study to the case of Euclidean matching, in which the distance ℓ_{ij} is a function of the positions of the points i and j in Euclidean space. In this case the various distances are obviously correlated (e.g. through triangular inequalities).

Let us first recall the results which have been obtained on the random link model. If $\rho(\ell)$ is the distribution of lengths of the links, the only important feature of ρ is its behaviour around $\ell = 0$ (we suppose for definiteness that $\ell = 0$ is the smallest possible distance). If :

$$\rho(\ell) \overset{\ell \to 0}{\sim} \ell^r / r! \tag{2}$$

Then the length of the optimal matching has been found to have the following behaviour in the limit where the number of points, $2N$, goes to infinity :

$$L_r(N) \overset{N \to \infty}{\sim} N^{1 - \frac{1}{r+1}} \hat{L}_r \tag{3}$$

where :

$$\hat{L}_0 = \frac{\pi^2}{12} ; \quad \hat{L}_1 = 1.144 ; \quad \hat{L}_2 = 1.5197 ; \ \dots \tag{4}$$

These results have been obtained with the replica/cavity method [3, 4], with a replica symmetric Ansatz. Furthermore this Ansatz has been shown to be stable for $r = 0$ [5], and the corresponding results in the case of bipartite matching with $r = 0$ and $r = 1$ are in good agreement with numerical simulations [5].

As we have suggested before [4], these results can be used as approximants to the Euclidean matching problem. It turns out that it is useful to introduce a generalized Euclidean problem as follows. The points are chosen randomly (with uniform distribution) inside a hypercube of side 1 in a D dimensional space. If $\mathbf{x}_i$ and $\mathbf{x}_j$ are two such points, we define the ν-distance between them by :

$$\ell_{ij} = |\mathbf{x}_i - \mathbf{x}_j|^{\nu} . \tag{5}$$

Finding the optimal matching with this matrix of distances will be called the ν-Euclidean matching problem. The usual Euclidean case is of course recovered for $\nu = 1$.

For this kind of matrix of distances, one expects that in the large N limit the only relevant links are short ones. Their length should scale as the distance between near neighbours, i.e. $N^{-\nu/D}$. Hence one expects a scaling of the length of the optimal matching :

$$L_{D,\nu}(N) \overset{N \to \infty}{\sim} N^{1-\frac{\nu}{D}} \tilde{L}_{D,\nu} \tag{6}$$

(remember that N is the number of bounds in the matching i.e. half the number of points).

In order to see the connection between the ν-Euclidean problem and the random link problem, let us give the distribution of ν-distances for the relevant short links :

$$\rho_{D,\nu}(\ell) \sim \frac{S_D}{\nu} \ell^{D/\nu - 1} \tag{7}$$

where $S_D = 2\pi^{D/2}(\Gamma(D/2))^{-1}$ is the surface of the D dimensional unit sphere. Comparing the two distributions of links (2) and (7), one can guess that the random link problem defined by parameter r will be related to the ν-Euclidean problem in D dimensions, whenever the following relation holds :

$$\nu = \frac{D}{r+1} . \tag{8}$$

In this note we shall prove that the Euclidean correlations can be neglected in the limit $\nu \to \infty$, $D \to \infty$, $\frac{D}{\nu} = r + 1$ fixed (F1). The length of the optimal matching is then just given by the random link result, apart from a trivial change of scale which can be read from (2) and (7).

This change of scale is characterized by the parameter :

$$\alpha_{D,\nu} = \left(\frac{\nu}{S_D \Gamma\left(\frac{D}{\nu}\right)} \right)^{\frac{\nu}{D}} = \left(\frac{\nu \Gamma\left(\frac{D}{2}\right)}{2\pi^{D/2} \Gamma\left(\frac{D}{\nu}\right)} \right)^{\nu/D} \tag{9}$$

and the vanishing of the effect of Euclidean correlations at large D simply means that the rescaled length :

$$\ell_{D,\nu} = \frac{\tilde{L}_{D,\nu}}{\alpha_{D,\nu}} \tag{10}$$

tends towards the random link result $\hat{L}_{r = \frac{D}{\nu} - 1}$ (defined in (3)), in the limit where $D \to \infty$, $\nu \to \infty$, $\frac{D}{\nu}$ fixed.

Of course Euclidean correlations induce some corrections to this formula in finite D. Hereafter we shall show how these can be incorporated in the computation in a systematic way. We shall give explicit results for $\ell_{D,\nu}$ and $\tilde{L}_{D,\nu}$ including triangular correlations (and neglecting higher order connected correlation functions). These results are contained in the table at the end of this paper. For the *real* Euclidean problem ($\nu = 1$), in D dimensions, the asymptotic length of the optimal matching scales as :

$$L_{D,1}(N) \overset{N \to \infty}{\sim} N^{1-\frac{1}{D}} \ell_{D,1} \alpha_{D,1} \tag{11}$$

where the series $\ell_{D', \frac{D'}{D}}$; $D' = D, D+1, \ldots$, interpolates between the real result $\ell_{D,1}$, ($D' = D$), and the random link result $\hat{L}_{D-1}$, ($D' \to \infty$).

Let us now turn to the computation. The starting point follows reference [3, 6]. We define the partition function as :

$$Z = \sum_{\{n_{ij} = 0,1\}} \prod_i \delta\left(\sum_j n_{ij} - 1\right) \exp\left(-\beta N^{\frac{\nu}{D}} \sum_{i<j} n_{ij} \ell_{ij}\right) \tag{12}$$

(the factor $N^{\frac{\nu}{D}}$ is the correct scaling of temperature which ensures that a good thermodynamic limit exists

[7]). In order to compute the quenched average of log Z over the distribution of distances, $\overline{\log Z}$, (where the bar denotes the average over the different realizations of the ℓ's) we use the replica method [1]. Introducing exponential representations of the δ functions we get :

$$\overline{Z^n} = \prod_{j=1}^{N} \prod_{a=1}^{n} \left(\int_0^{2\pi} \frac{d\lambda_j^a}{2\pi} e^{i\lambda_j^a} \right) \prod_{i<j} \prod_{a=1}^{n} \left[1 + e^{-\beta N^{\frac{\nu}{D}} \ell_{ij}} e^{-i(\lambda_i^a + \lambda_j^a)} \right]. \tag{13}$$

For each link $i<j$ we can expand the last product :

$$\prod_{a=1}^{n} \left[1 + e^{-\beta N^{\frac{\nu}{D}} \ell_{ij}} e^{-i(\lambda_i^a + \lambda_j^a)} \right] = 1 + u_{ij} ,$$

$$u_{ij} \equiv e^{-\beta N^{\frac{\nu}{D}} \ell_{ij}} \sum_a e^{-i(\lambda_i^a + \lambda_j^a)} + e^{-2\beta N^{\frac{\nu}{D}} \ell_{ij}} \sum_{a<b} e^{-i(\lambda_i^a + \lambda_j^a + \lambda_i^b + \lambda_j^b)} + \cdots \tag{14}$$

so that :

$$\overline{Z^n} = \prod_{j=1}^{N} \prod_{a=1}^{n} \left(\int_0^{2\pi} \frac{d\lambda_j^a}{2\pi} e^{i\lambda_j^a} \right) \left[1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum'_{(j_1 k_1), \ldots, (j_m k_m)} \overline{u_{j_1 k_1} \cdots u_{j_m k_m}} \right] \tag{15}$$

where (jk) denotes the (unoriented) link between j and k, and $\sum'$ means the sum over all m-uples of distinct links.

To proceed we need to perform the quenched average over the link distribution. In the random link problem the u_{ij} on various links are uncorrelated ; everything is then expressed in terms of [3] :

$$\overline{u_{ij}} = \frac{1}{N} \sum_{p=1}^{\infty} \frac{1}{\beta^{D/\nu}} \int_0^{\infty} d\ell \, \rho_{D,\nu}(\ell) \, e^{-p\ell} \sum_{1 \leq a_1 < \cdots < a_p \leq n} e^{-i(\lambda_i^{a_1} + \cdots + \lambda_i^{a_p}) - i(\lambda_j^{a_1} + \cdots + \lambda_j^{a_p})} . \tag{16}$$

The series in (15) is then easily exponentiated and gives :

$$\overline{Z^n} = \prod_{j=1}^{N} \prod_{a=1}^{n} \left(\int_0^{2\pi} \frac{d\lambda_j^a}{2\pi} e^{i\lambda_j^a} \right) e^{\sum_{i<j} \overline{u_{ij}}} . \tag{17}$$

Equation (17) was the starting point of our previous computation [3]. Euclidean correlations of course prevents the m link average $\overline{u_{j_1 k_1} \cdots u_{j_m k_m}}$ in (15) from factorizing in general. However the two link average factorizes and the first correlations appear at the level of three link correlations, which are non zero whenever the three links build a triangle. We shall express these correlations in detail hereafter, but for the present purpose it is enough to know the way it scales with N. In the same way as the factor $1/N$ in (16) reflects the fact that the probability of finding a short link (of length $< N^{-\nu/D}$) is $\sim \frac{1}{N}$, it is easy to see that the probability that the three links in a triangle be short is $\sim \frac{1}{N^2}$. (Because of the triangular inequality, we need to ensure that two of the links be short, and the third one will automatically be short also). Therefore the connected three link average in a triangle scales as :

$$\overline{u_{ij} u_{jk} u_{ki}}^C \equiv \overline{u_{ij} u_{jk} u_{ki}} - \overline{u_{ij}}\, \overline{u_{jk}}\, \overline{u_{ki}} \sim 1/N^2 . \tag{18}$$

So if we neglect the higher order connected link averages we find after exponentiation of the series in (15) :

$$\overline{Z^n} = \prod_{j=1}^{N} \prod_{a=1}^{n} \left(\int_0^{2\pi} \frac{d\lambda_j^a}{2\pi} e^{i\lambda_j^a} \right) e^{\sum_{(ij)} \overline{u_{ij}} + \frac{1}{3!} \sum'_{(ij)(k\ell)(mn)} \overline{u_{ij} u_{k\ell} u_{mn}}^C} . \tag{19}$$

It is worth noticing that both terms in the exponent of (19) are of order N as they should : the first term has $\overline{u_{ij}} \sim 1/N$ and a summation over N^2 links. In the second term the only non vanishing terms are these in which (ij), $(k\ell)$, (mn) build up a triangle ; there are N^3 such terms, each of them being of order $1/N^2$ from (18). Higher order connected link averages could be included in (19) as well, but in this paper we shall only consider the effect of triangular correlations in order to keep the computations (relatively) simple.

We must now find the expression of triangular correlations for ν-distances in a D dimensional Euclidean space. Taking three points at random the probability that the corresponding triangle will have sides of ν-lengths equal to ℓ, ℓ', ℓ'' is :

$$\mathcal{T}_{D,\nu}(\ell, \ell', \ell'') = \int \mathrm{d}^D\mathbf{x}\, \mathrm{d}^D\mathbf{y}\, \delta[\ell - |\mathbf{x}|^\nu]\, \delta[\ell' - |\mathbf{y}|^\nu]\, \delta[\ell'' - |\mathbf{x}-\mathbf{y}|^\nu]\,. \tag{20}$$

In the relevant limit of short links (remember that the occupied links in the matching will be of length $N^{-\frac{\nu}{D}}$) a careful computation gives :

$$\mathcal{T}_{D,\nu}(\ell, \ell', \ell'') = \frac{S_D S_{D-1}}{\nu^3 2^{D-3}} (\ell\, \ell'\, \ell'')^{\frac{2}{\nu}-1} [A(\ell, \ell', \ell'')]^{\frac{D-3}{2}} \,.\, \theta[A(\ell, \ell', \ell'')]\, \theta(\ell)\, \theta(\ell')\, \theta(\ell'') \tag{21}$$

where :

$$A(\ell, \ell', \ell'') = 2(\ell^{2/\nu} \ell'^{2/\nu} + \ell'^{2/\nu} \ell''^{2/\nu} + \ell''^{2/\nu} \ell^{2/\nu}) - (\ell^{4/\nu} + \ell'^{4/\nu} + \ell''^{4/\nu}) \tag{22}$$

(one can check that $\int \mathrm{d}\ell''\, \rho_{D,\nu}(\ell, \ell', \ell'') = \rho_{D,\nu}(\ell)\, \rho_{D,\nu}(\ell')$: the two link connected average vanishes).

From (16) and (19), the typical quantities which we must compute are :

$$\iiint \mathrm{d}\ell\, \mathrm{d}\ell'\, \mathrm{d}\ell'' [\mathcal{T}_{D,\nu}(\ell, \ell', \ell'') - \rho_{D,\nu}(\ell)\, \rho_{D,\nu}(\ell')\, \rho_{D,\nu}(\ell'')]\, \mathrm{e}^{-\beta N^{\frac{\nu}{D}}(p\ell + p'\ell' + p''\ell'')}$$

$$\begin{cases} = 0, \text{ if } p\,p'\,p'' = 0 \\ = \dfrac{1}{N^2} \dfrac{1}{\beta^{2D/\nu}} \displaystyle\iiint_0^\infty \mathcal{T}_{D,\nu}(\ell, \ell', \ell'')\, \mathrm{e}^{-(p\ell + p'\ell' + p''\ell'')} + \mathcal{O}\left(\frac{1}{N^3}\right) \quad \text{if } p \geq 1 \text{ and } p' \geq 1 \text{ and } p'' \geq 1\,. \end{cases} \tag{23}$$

Using (19) and (23) we can derive the expression for $\overline{Z^n}$ including triangular correlations. It is useful to introduce the order parameter :

$$\frac{1}{2N} \sum_{j=1}^{2N} \mathrm{e}^{-i(\lambda_j^{a_1} + \cdots + \lambda_j^{a_k})} \equiv Q_{a_1 \ldots a_k} \qquad \begin{matrix} 1 \leq k \leq n \\ 1 \leq a_1 < \cdots < a_k \leq n \end{matrix} \tag{24}$$

and to enforce these equalities through Lagrange multipliers $\hat{Q}_{a_1 \ldots a_k}$.

One gets :

$$\overline{Z^n} = \prod_{k=1}^{n} \prod_{a_1 < \cdots < a_k} \left(\int_{-\infty}^{+\infty} \mathrm{d}Q_{a_1 \ldots a_k} \int_{-i\infty}^{+i\infty} \mathrm{d}\hat{Q}_{a_1 \ldots a_k} \right) \mathrm{e}^{N(G_1(Q,\hat{Q}) + G_3(Q,\hat{Q}))} \tag{25}$$

where G_1 is the usual contribution of the random link model :

$$G_1 = -2 \sum_{k=1}^{n} \sum_{a_1 < \cdots < a_k} \hat{Q}_{a_1 \ldots a_k} Q_{a_1 \ldots a_k} + 2 \sum_{k=1}^{n} \frac{1}{\beta^{D/\nu}} \int_0^\infty \rho_{D,\nu}(\ell)\, \mathrm{e}^{-k\ell} \sum_{a_1 < \cdots < a_k} Q^2_{a_1 \ldots a_k} +$$
$$+ 2\, Ln \left[\prod_{a=1}^{n} \left(\int_0^{2\pi} \frac{\mathrm{d}\lambda^\alpha}{2\pi} \mathrm{e}^{i\lambda^a} \right) \exp\left(\sum_{k=1}^{n} \sum_{a_1 < \cdots < a_k} \hat{Q}_{a_1 \ldots a_k} \mathrm{e}^{-i(\lambda^{a_1} + \cdots + \lambda^{a_k})} \right) \right] \tag{26}$$

and G_3 is the new term coming from the triangular correlations :

$$G_3 = \frac{4}{3} \sum_{k,k',k''=1}^{n} \frac{1}{\beta^{2D/\nu}} \iiint \mathrm{d}\ell\, \mathrm{d}\ell'\, \mathrm{d}\ell''\, \mathcal{T}_{D,\nu}(\ell, \ell', \ell'')\, \mathrm{e}^{-k\ell - k'\ell' - k''\ell''} \times$$
$$\times \frac{1}{k!\, k'!\, k''!} \sum'_{a_1 \ldots a_k b_1 \ldots b_{k'} c_1 \ldots c_{k''}} Q_{a_1 \ldots a_k b_1 \ldots b_{k'}} Q_{b_1 \ldots b_{k'} c_1 \ldots c_{k''}} Q_{c_1 \ldots c_{k''} a_1 \ldots a_k} \tag{27}$$

where the $\sum'$ means that all the replica indices must be distinct one from another and the Q's are symmetric under the permutations of their indices.

We can now compute $\overline{Z^n}$ using a saddle point method. In this paper we shall look only for a replica symmetric saddle point :

$$Q_{a_1 \dots a_p} = Q_p \; ; \quad \hat{Q}_{a_1 \dots a_p} = \hat{Q}_p \, , \qquad p = 1, \dots, n \, . \tag{28}$$

Two approaches are possible : one can treat G_3 as a small perturbation to G_1 (i.e. compute G_3 at the saddle point given by G_1), or one can solve directly the saddle point equation including both G_1 and G_3. Here we describe only the second approach which is more general than the first one and not much more complex.

As in the random link case [3] the saddle point equations are more easily written in terms of a generating function of the $\hat{Q}_p$. We define :

$$G(x) \equiv \sum_{p=1}^{\infty} (-1)^{p-1} \frac{\hat{Q}_p}{p!} e^{p\alpha\beta x} \tag{29}$$

where the parameter α is the change of length scale $\alpha_{D,\nu}$ introduced in (9). In order to lighten the notation we shall not write explicitly its indices D, ν in the following computations.

The factor $\alpha\beta x$ in the definition of $G(x)$ has been chosen in a way such as to insure that G will have a limit when $\beta \to \infty$ or when $D \to \infty$ (at fixed $\frac{D}{\nu}$). From the saddle point equations, using the same methods of reference [3], a long, but straightforward computation leads to the following integral equation for $G(x)$:

$$\begin{aligned} G(x) = {} & 2 \int dy \, e^{-G(y)} \int_0^{\infty} d\ell \, \ell^{\frac{D}{\nu} - 1} \, \Gamma\left(\frac{D}{\nu}\right)^{-1} \frac{\partial}{\partial x} I[\alpha\beta(x + y - \ell)] + \\ & + 2\alpha^{2D/\nu} \int dy \, G'(y) \, e^{-G(y)} \int dz \, G'(z) \, e^{-G(z)} \\ & \iiint d\ell \, d\ell' \, d\ell'' \, P_{D,\nu}(\ell, \ell', \ell'') \frac{1}{\alpha\beta} \frac{\partial}{\partial x} K[\alpha\beta(y + z - \ell), \alpha\beta(x + z - \ell'), \alpha\beta(x + y - \ell'')] \end{aligned} \tag{30}$$

where the integration kernels are :

$$\begin{aligned} I(u) &= \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(p!)^2} e^{pu} \\ K(u, u', u'') &= \sum_{p,p',p''=1}^{\infty} \frac{(-1)^{p+p'+p''-1} (p + p' + p'' - 1)! \, e^{pu + p'u' + p''u''}}{p! \, p'! \, p''! \, (p + p')! \, (p' + p'')! \, (p'' + p)!} \, . \end{aligned} \tag{31}$$

The free energy (on the saddle point) is then expressed in terms of the function $G(x)$ solution of (30) as :

$$\begin{aligned} F &= -\frac{1}{Nn\beta} \operatorname{Log} \overline{Z^n} \\ &= \alpha \int dy \, G(y) \, e^{-G(y)} - 2\alpha \int dy [e^{-e^{\alpha\beta y}} - e^{-G(y)}] + \\ & \quad + \frac{2}{3} \alpha^{(1 + 2D/\nu)} \int dx \int dy \int dz \, G'(x) \, e^{-G(x)} \, G'(y) \, e^{-G(y)} \, G'(z) \, e^{-G(z)} \\ & \quad \iiint d\ell \, d\ell' \, d\ell'' \, P_{D,\nu}(\ell, \ell', \ell'') \frac{1}{\alpha\beta} K[\alpha\beta(y + z - \ell), \alpha\beta(x + z - \ell'), \alpha\beta(x + y - \ell'')] \, . \end{aligned} \tag{32}$$

The above expressions become simpler in the interesting zero temperature limit $\beta \to \infty$. In order to take this limit one needs the asymptotic behaviour of the kernels I and K. After some work one finds :

$$\begin{aligned} I(\alpha\beta u) & \overset{\alpha\beta \to \infty}{\sim} \theta(u) \\ K(\alpha\beta u, \alpha\beta u', \alpha\beta u'') & \overset{\alpha\beta \to \infty}{\sim} \alpha\beta \, \theta(u) \, \theta(u') \, \theta(u'') \inf(u, u', u'') \end{aligned} \tag{33}$$

where θ is the usual step function. Then the saddle point equation becomes :

$$G(x) = 2 \int_0^\infty \mathrm{d}\ell \, \rho_{D,\nu}(\ell) \, \mathrm{e}^{-G(\ell - x)} + 2 \, \alpha^{\frac{2D}{\nu}} \int \mathrm{d}\ell \int \mathrm{d}\ell' \int \mathrm{d}\ell'' \, \mathcal{T}_{D,\nu}(\ell, \ell', \ell'') \, L(x, \ell, \ell', \ell'') \quad (34)$$

where L is given by :

$$L(x, \ell, \ell', \ell'') = \Big[\theta(2x + \ell - \ell' - \ell'')\{\mathrm{e}^{-G(x+\ell-\ell'')-G(\ell''-x)} - \mathrm{e}^{-G(x+\ell-\ell')-G(x+\ell-\ell'')}\} + \\ + \int_{\mathrm{Max}\,(\ell''-x,\, x+\ell-\ell')}^{\infty} \mathrm{d}y (G'(y) \, \mathrm{e}^{-G(y)} \, . \, \mathrm{e}^{-G(y+\ell'-\ell'')})\Big] + [\ell' \rightleftarrows \ell''] \quad (35)$$

where the last term is obtained from the expression between brackets through the permutation of ℓ' and ℓ''. The corresponding limiting value of the free energy for $\beta \to \infty$, which is the ground state energy $\tilde{L}_{D,\nu}$ defined in (6), is given by :

$$\tilde{L}_{D,\nu} = \alpha \left\{ \frac{4}{3} \int \mathrm{d}y \, G(y) \, \mathrm{e}^{-G(y)} - 2 \int \mathrm{d}y [\theta(y) - \mathrm{e}^{-G(y)}] - \frac{2}{3} \int \mathrm{d}y \, \mathrm{e}^{-G(y)} \int_0^\infty \mathrm{d}\ell \, \rho_{D,\nu}(\ell) \, \mathrm{e}^{-G(\ell - y)} \right\}. \quad (36)$$

So in order to solve the matching problem including the triangular correlations for a ν-Euclidean problem in D dimensions, we must first find the function $G(x)$ solution of (34), and then compute the ground state energy through (36).

Just for comparison, the approach in which G_3 is treated as a small perturbation to G_1 gives :

$$\tilde{L}_{D,\nu} = \alpha \Bigg\{ \int \mathrm{d}y \, G(y) \, \mathrm{e}^{-G(y)} - 2 \int \mathrm{d}y [\theta(y) - \mathrm{e}^{-G(y)}] - \\ - \frac{4}{3} \alpha^{\frac{2D}{\nu}} \iiint \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, G'(x) \, \mathrm{e}^{-G(x)} \, G'(y) \, \mathrm{e}^{-G(y)} \, G'(z) \, \mathrm{e}^{-G(z)} \\ \times \iiint \mathrm{d}\ell \, \mathrm{d}\ell' \, \mathrm{d}\ell'' \, \mathcal{T}_{D,\nu}(\ell, \ell', \ell'') \frac{1}{\alpha\beta} K[\alpha\beta(y + z - \ell), \alpha\beta(x + z - \ell'), \alpha\beta(x + y - \ell'')] \Bigg\} \quad (37)$$

where $G(x)$ is the solution of the saddle point equation computed in absence of the G_3 term.

One important point is that the triangular correlations (last term in (34)) become irrelevant in the limit $\nu \to \infty$, $D \to \infty$, $\frac{D}{\nu} = r + 1$ fixed. This can be seen by explicit majoration, under the assumption that the asymptotic behaviour of $G(x)$ is not too much perturbed by these new correlations. This proves the announced asymptotic result :

$$\lim_{D \to \infty,\, \nu \to \infty,\, \frac{D}{\nu}\text{ fixed}} \left(\ell_{D,\nu} \equiv \frac{\tilde{L}_{D,\nu}}{\alpha_{D,\nu}} \right) = \hat{L}_{\frac{D}{\nu}-1} \, . \quad (38)$$

In order to go beyond the random link model and incorporate some Euclidean correlations, we have solved the saddle point equation (34) by discretizing the $G(x)$ function and using a simple iteration of (34), starting from the random link function $G(x)$. We can then evaluate $\tilde{L}_{D,\nu}$ in (36). The results for $\tilde{L}_{D,\nu}$ and the rescaled length $\ell_{D,\nu}$ (defined in (10)) are given in the table, for the cases $\frac{D}{\nu} = 2$ and 3 which would correspond, for the usual Euclidean problem ($\nu = 1$) to the matching problem in 2 and 3 dimensions. We see that the effect of triangular

Table I. — $\frac{D}{\nu} = 2$ ($r = 1$).

D	2	3	4	6	8	∞
$\ell_{D,\nu}$	1.07	1.06	1.07	1.09	1.11	1.144
$\alpha_{D,\nu}$	0.3989	0.3455	0.3183	0.3110	0.3510	
$\tilde{L}_{D,\nu}$	0.426	0.367	0.341	0.339	0.388	

Table II. — $\frac{D}{\nu} = 3$ ($r = 2$).

D	3	4	6	8	∞
$\ell_{D,\nu}$	1.45	1.46	1.47	1.49	1.51973
$\alpha_{D,\nu}$	0.3414	0.3232	0.3183	0.3450	
$\tilde{L}_{D,\nu}$	0.495	0.471	0.469	0.513	

correlations is rather small, especially for large values of D, as expected. The comparison of these results with numerical simulations will be performed in another paper.

In this paper we have shown how to include Euclidean correlations into the computations on the matching problem. We have carried out explicit computations including triangular correlations, but clearly higher order correlations can be incorporated in the same way, at the price of having to solve more and more complex saddle point equations. We hope that this kind of improvement will be generalized to other spin glass like systems like the travelling salesman problem (for which a possible solution has already been proposed in the random link case [8]), or to real Euclidean spin glasses. It is quite possible that the equivalent of the *random link case* could well be the mean field theory (SK model [9]), on which the present type of approach could build some improved solution for the finite dimensional case.

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Footnote.

It is amusing to remark that if we face the problem of connecting the points by a wireless bridge, the electric power needed to establish the connection is proportional to the power $D - 1$ of the distance and therefore in this case we should take ν equal to $D - 1$. In the unusual (at least for this problem) limit D going to infinity, we obtain the $r \doteq 0$ model.

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