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Mean-Field Equations for the Matching and the Travelling Salesman Problems.

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Abstract. – The matching problem and the travelling salesman problem are investigated in very high dimensions using mean-field equations valid for one given sample. The same results of the replica approach are consistently found. Some applications of these results to the two-dimensional case are briefly discussed.

Quite recently [1-3] the matching problem and the travelling salesman problem (in the case where the distances are random variables) have been studied using the replica approach in the thermodynamic limit (infinite number of points). The solution to both problems has been found under the hypothesis that the replica symmetry is not broken [1, 3].

In the nutshell if replica symmetry is not broken [4], the quasi-optimal solutions coincide with the optimal one; more precisely, if we consider a problem with N points (N large), we call a configuration quasi-optimal, if the relative difference of the cost functions of this configuration and of the optimal solution is proportional to $1/N$. The replica symmetry is broken if there is (with probability one for large N) a quasi-optimal configuration which differs from the optimal one in a number of points proportional to N.

Roughly speaking, if we neglect the relative differences in the cost functions of two configurations which are order $1/N$ and we consider that two configurations, which differ one from the other in a set of points whose number increases less than linearly in N, are identical, then the replica symmetry is not broken if the optimal solution is unique. More detailed investigations are needed to clarify the correctness of this crucial assumption.

The aim of this note is to show how the results of the replica approach can be obtained by studying simple mean-field equations. In this way we establish a probabilistic interpretation of the results of the replica approach.

We start by defining the two problems: in both cases we have N cities (points): $d_{i,k}$ is the cost for travelling between the city i and the city k (here we keep to the case where $d_{i,k}$ is a symmetric matrix). The $d_{i,k}$ are assumed to be independent random variables with a probability distribution $P(d)$ such that

$$P(d) \approx d^r/r! \quad \text{for } d \to 0 . \tag{1}$$

Each choice of the $d_{i,k}$ is an instance of the problem. A configuration of the system is a set of $N/2$ (N must be even) links which connect pairwise all points for the matching problem or a closed line which goes through all the points (this line contains N links) in the case of the travelling salesman problem. The cost (or energy) of a configuration $H(C)$ will be the sum of the cost of all the links (in order to lighten the notation we have not indicated the dependence of $H(C)$ on the d's).

In both cases we can define a partition function Z and a free energy F (at given $d_{i,k}$) in the usual way:

$$Z[d,\beta] = \sum_{\{C\}} \exp[-\beta N^{\delta} H(C)] , \qquad F[d,\beta] = -(\beta N)^{-1} \ln Z , \tag{2}$$

where the sum is done over all configurations C and $\delta = (1+r)^{-1}$. We notice that δ has been chosen in such a way [5] that F is of $O(1)$ when $N \to \infty$ and it depends on β in a nontrivial way.

We want to compute the value $F(\beta)$ of the average (on all the possible choices of the distances $d_{i,k}$ according to the probability distribution equation (1)) of the free energy $F[d, \beta]$ in the limit $N \to \infty$. It is believed that for a generic instance and for a sufficiently large value of N:

$$\lim_{\beta \to \infty} F(\beta) = \min_C N^{(\delta-1)} H(C) , \tag{3}$$

so that $F(\infty)$ is related to the cost of the optimal configuration (a well-known optimization problem).

In ref. [1, 3] the problem was solved (under the hypothesis of no replica symmetry breaking) by introducing a set of auxiliary quantities Q_p (p going from 1 to ∞). An auxiliary free energy $F(Q, \beta)$ can be written as function of the Q's and the value of the free energy $F(\beta)$ is the value of $F(Q, \beta)$ at the point where it is stationary with respect to the Q's. In other words we have to solve the equations

$$\partial F / \partial Q_k = 0 , \tag{4}$$

and evaluate $F(Q, \beta)$ at the solution of eq. (4).

Here we use a different approach: we want to derive mean-field equations (of the TAP type [6]) and solve them under the hypothesis of the uniqueness of the thermodynamic state. In spin glasses this approach gives the same results as the replica approach based on no replica symmetry breaking. In this respect we have been strongly stimulated by ref. [7], although our approach is technically different from the one of this last paper.

It is convenient to consider an auxiliary model in which a m-component spin σ_i^a ($i = 1, N$, $a = 1, m$) of modulus m($\sum_{a=1,m} (\sigma_i^a)^2 = m$) sits at each point i (this procedure is a standard tool in the study of self-avoiding walks or polymers [8]). The partition function is defined to be

$$Z = \sum_{\{\sigma\}} \Big\{ \prod_{j=1,N} (1 + h\sigma_j^1) \exp[\gamma/2 \sum_{i,k=1,N} \sum_{a=1,m} R_{i,k} \sigma_i^a \sigma_k^a] \Big\} , \qquad R_{i,k} = \exp[-\beta N^{\delta} d_{i,k}] , \quad R_{i,i} = 0 , \tag{5}$$

where h is like an external magnetic field.

It is interesting to consider this model in the limit m going to zero. We use the well-known fact that the partition function of a m-component spin model can be written in the limit $m \to 0$ as a sum of contributions of self-avoiding paths.

At $h = 0$, $(Z - 1)/m$ is the sum over all possible closed self-avoiding loops (only one loop at a time) of the quantity

$$\gamma^L \prod_{i,k} R_{i,k} , \tag{6}$$

where L is the length of the loop and the products is done only on the (ordered) pairs i, k which belong to the loop. It is evident that when $L = N$ the weight (6) of the loop is proportional to the contribution of the same loop to the partition function of the travelling salesman.

For $h \neq 0$, in the limit $m \to 0$, the partition function is given by the sum over all open self-avoiding paths (K paths may be present simultaneously) with the same weight as in eq. (6) plus an additional factor h^{2K}.

It is easy now to check that in the limit γ going to infinity (at zero h) we recover (from $(Z - 1)/m$) the partition function of the travelling salesman, while in the limit h going to infinity (at nonzero γ) we recover (from Z) the matching problem (if we send γ to infinity at a fixed value of $\alpha \equiv h^2/\gamma$, we obtain a model whose free energy interpolates between the travelling salesman ($\alpha \to \infty$) and the matching ($\alpha \to 0$)). We can now write the generalized TAP equations for the model and later we will specialize them to the two different cases.

Let us suppose that we have a spin model with N sites with the partition function (5), such that

$$\langle \sigma_i^1 \rangle = m_i , \tag{7}$$

the other components of σ have zero expectation values. We also assume that at a given temperature there is only one equilibrium state (no breaking of the replica symmetry and the connected correlation functions are so small (*e.g.* $O(1/N)$) that they can be neglected.

The strategy consists in comparing the properties of two systems of N and $N+1$ spins which have N spins in common and imposing that the properties of the last system coincide with that of the first one [9].

Under these assumptions, if we add a new site to the N-sites system, it is easy to see that generalized Bethe formulae are valid, in particular we find that the magnetization of the new spin (*i.e.* the spin at $N+1$, m_{N+1}) can be written as a function of the costs for going from the $(N+1)$-th site to each of the others N sites and of the magnetizations of the others N sites of the system before the new spin has been added (these magnetizations will be called «cavity magnetizations» and they will be denoted as m_i^c, $i = 1, N$).

For generic m the formulae would be very complicated; fortunately strong simplifications are present in the limit $m \to 0$: they are mainly due to the relation $(\sigma_i^1)^k = 0$ for $k > 2$; in other words σ_i^1 is a nihilpotent quantity (this result can be understood using the supersymmetric representation of the nonlinear σ-model for $m = 0$ [10]).

The results further simplify in the two limits ($h \to \infty$, or $\gamma \to \infty$ at $h = 0$); we finally find after an appropriate rescaling of the magnetization (from now on we have that $m_i = \langle \sigma_i^1 \rangle / h^{1/2}$ and $m_i = \langle \sigma_i^1 \rangle / \gamma^{1/2}$, respectively, in the matching and in the travelling salesman cases):

$$\begin{cases} m_{N+1} = \left(\sum_{i=1,N} R_{N+1,i} m_i^c \right)^{-1} & \text{(matching)} , \\ m_{N+1} = \left(\sum_{i=1,N} R_{N+1,i} m_i^c \right) / \left(\left(\sum_{1 \le i < k \le N} R_{N+1,i} m_i^c R_{N+1,k} m_k^c \right) \right. & \text{(travelling salesman)} . \end{cases} \tag{8}$$

We also find that the probability ($n_{N+1,i}$) that the link from $N+1$ to i is occupied, is given by

$$\begin{cases} n_{N+1,i} = R_{N+1,i}\, m_i^c / \Big(\sum\limits_{k=1,N} R_{N+1,k}\, m_k^c \Big) & \text{(matching)}, \\ n_{N+1,i} = R_{N+1,i}\, m_i^c \Big(\sum\limits_{j=1,N;J\neq i} R_{N+1,k}\, m_k^c \Big) \Big(\sum\limits_{1\leq j<k\leq N} R_{N+1,j}\, m_j^c\, R_{N+1,k}\, m_k^c \Big)^{-1} & \text{(travelling salesman)}. \end{cases} \tag{9}$$

After sample averaging the probability distribution of the magnetization in the system with N or $N+1$ sites must be the same; if we recall that the magnetizations are independent random variables with probability distribution $P(m)$, we easily see that eq. (8) implies a complicated integral equation for the function $P(m)$, which we do not dare to write down explicitly. In this way the model is solved and the probability distribution $P(m)$ can be found numerically by iterating the integral equation for the probability distribution which is induced by eq. (8). Equation (9) can be used to compute the expectation value of the cost function.

It is interesting to see the connection with the replica approach: if we define the quantity Q_k as

$$Q_k = (\beta k)^{-(r+1)} \int dm\, P(m)\, m^k , \tag{10}$$

it is possible to check (in full generality for the matching problem, in many particular cases for the travelling salesman problem) that the Q's must satisfy the equations which were found in the replica approach. In this way we have obtained a direct derivation of the results of the replica approach.

Altough the analytic treatment of the model can be done in the best way in the cavity approach where the magnetization at the site $N+1$ is computed in terms of the magnetizations of the other N sites when the spin at site $N+1$ is removed, it is interesting to write down the equations in terms of the magnetizations after the site $N+1$ is added to the system; in this way we obtain some kind of generalized TAP equations.

If we compute the magnetizations (m_i) of one of the N sites (*e.g.* i) in the presence of the new site as a function of the (cavity) magnetizations (*i.e.* m_j^c, $j=1, N$) of the N sites before the new site is added, we find that it can be written in the matching case (for the points i such that $R_{N+1,i}$ is of order 1, not of order $1/N$ or smaller) as

$$m_i = m_i^c (1 - n_{N+1,i}) , \tag{11}$$

where $n_{N+1,i}$ is given by eq. (9). It is evident that the relative difference between m_i and m_i^c is very small in most of the cases, but in some cases it will be very important; therefore, we cannot, as in the spin glass case, perform a perturbative expansion in this difference.

If now we compute the n's as functions of the m's we have a closed set of equations in which there is no reference to the cavity fields which have been used to derive them; we finally get

$$n_{i,k}(1 - n_{i,k}) = m_i R_{i,k} m_k , \qquad (m_i)^{-1} = \Big(\sum_{k=1,N;k\neq i} R_{i,k}\, m_k / (1 - n_{i,k}) \Big) . \tag{12}$$

In the case of the travelling salesman the situation is more complex; if we try to obtain the generalization of eq. (11) to this case, it turns out that also the quantities q_i defined by

$\langle(\sigma_i^1)^2\rangle$ are important here. After some work one finds that

$$\begin{cases} q_i = q_i^c(1-n_{N+1,i})\,, \\ m_i = m_i^c(1-n_{N+1,i}) + \dfrac{q_i^c}{m_i^c}\,n_{N+1,i}\,, \\ n_{N+1,i} = m_{N+1}R_{N+1,i}\,m_i^c - q_{N+1}(R_{N+1,i}\,m_i^c)^2\,, \\ q_{N+1} = [\sum\limits_{1\le j<k\le N} R_{N+1,j}\,m_j^c\,R_{N+1,k}\,m_k^c]^{-1}\,, \\ m_{N+1} = q_{N+1}\sum\limits_{i=1}^{N} R_{N+1,i}\,m_i^c \quad \text{(see (8))}\,. \end{cases} \tag{13}$$

The elimination of the cavity expectation values (m_i^c) is not easy here. However, by studying carefully what happens when two spins are added together to the system, we find the very interesting formula

$$m_j^{c(i)} = [R_{ji}\,q_i\,m_i + (1-n_{ij})\,m_j][(1-n_{ij})^2 - (R_{ij})^2 q_i q_j]^{-1}\,, \tag{14}$$

where $m_k^{c(i)}$ is the expectation value of σ_k^1 in the absence of the spin at the site i.

The matrix $n_{i,k}$ does not look symmetric, however it has been derived starting from the symmetric relation

$$n_{i,k} = m_i^{c(k)}\,R_i\,m_k^{c(i)}/(1 + m_i^{c(k)}R_{i,k}\,m_k^{c(i)})\,. \tag{15}$$

It is certainly interesting to investigate analytically and numerically the question of how many solutions does eq. (12) have and which are the properties of these solutions. As these equations are valid for one given sample, they can also be used to build up new algorithms to solve these problems. Work is in progress in these lines and it will be published elsewhere.

We conclude by shortly discussing the relevance of these findings to the Euclidean matching and travelling salesman problem in which the points are randomly chosen in real space (for simplicity in a hypercube of side 1) and the cost for travelling from one point (i) to an other point (k) depends on the usual Euclidean distance $l_{i,k}$. For example, we can consider the model where $d_{i,k} = (l_{i,k})^{D/(r+1)}$, D being equal to the dimensions of the space and r being a parameter characterizing the model. The most studied model in the literature corresponds to $D=2$, $r=1$.

An elementary computation shows that if we neglect the correlations between three or more distances (two distances are always uncorrelated), this model coincides (apart from a trivial rescaling) with the one considered in this paper (an intuitive argument, which we have not fully checked, suggests that the effect of the correlations is negligible in the limit $D\to\infty$ at fixed r and something similar to an $1/D$ expansion should be possible). Under the bold assumption that the effect of the correlations can be neglected in two dimensions (and the more subtle assumption of absence of replica symmetry breaking) the results of ref. [1, 3], imply that $F(\infty)$ (which is related to the total cost by eq. (3)) is equal to:

$$\begin{cases} \text{Matching:} & \pi/12 \;\; (r=0)\,, \qquad 0.323 \;\; (r=1)\,, \\ \text{Travelling salesman:} & 0.66 (r=0)\,. \end{cases} \tag{16}$$

The number for the matching with $r=1$ compares well with the numerical data: 0.322 (the numbers for $r=0$ are unfortunately unavailable).

We finally remark that another interesting model which is a generalization of the usual matching problem is the following: each city must be connected to other K cities and we want to minimize the total cost of connections; for $K=1$ we recover the usual matching.

If we stick our attention to this generalized matching problem for $K=2$, we can distinguish among two cases: a city can (case a)) or cannot (case b)) be connected twice to the same city. It is possible to prove in the replica approach (and also a direct argument can be constructed) that, when the correlations among the distances (and consequently among the costs) are neglected, the free energy of this generalized matching problem for $K=2$ is (apart a factor 2) the same of the matching (in case a)) or the same as the travelling salesman model (in case b)).

It seems that the most interesting models of this class are the generalized matching in which a city may be connected only once to the same city and consequently it must be connected to K different cities. It is very easy to apply our approach to this case; for $K=3$ we consider the following partition function:

$$Z = \sum_{\{\sigma,\tau,\lambda\}} \{ \prod_{j=1,N} \sigma_j^1 \tau_j^1 \lambda_j^1 \exp[\gamma/2 \sum_{i,k=1,N} R_{i,k} \sum_{a=1,m} [\sigma_i^a \sigma_k^a + \tau_i^a \tau_k^a + \lambda_i^a \lambda_k^a]]\} \qquad (17)$$

(a similar partition function can be written also for $K=2$).

Using the same technique as before, we readly obtain the following equation for $K=3$:

$$m_{N+1} = (\sum_{1\le i<k\le N} R_{N+1,i}\, m_i^{\mathrm{c}} R_{N+1,k}\, m_k^{\mathrm{c}}) (\sum_{1\le i<k<j\le N} R_{N+1,i}\, m_i^{\mathrm{c}} R_{N+1,k}\, m_k^{\mathrm{c}} R_{N+1,j}\, m_j^{\mathrm{c}})^{-1} . \qquad (18)$$

Similar equations can be obtained for higher values of K. A detailed study of these equations goes beyond the aims of this note.

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