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SK Model: The Replica Solution without Replicas.

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Abstract. – We introduce a new method, which does not use replicas, from which we recover all the results of the replica symmetry-breaking solution of the Sherrington-Kirkpatrick model.

Since its introduction in the context of spin glasses by EDWARDS and ANDERSON [1], the replica method has been carried to a high degree of sophistication. A solution of the mean-field theory (the SK model [2]) with replica symmetry breaking (RSB) has been proposed [3], and its physical meaning has been fully elucidated recently [4-7]. Although this replica solution builds up a coherent picture and provides us with a powerful method for analysing the equilibrium properties, it is difficult to put it on precise mathematical grounds.

In this paper we introduce an alternative method which does not rely on the replica trick, but leads to the same solution. It can be viewed as an analytic ansatz to solve the mean-field equation of TAP [8].

The basic idea is to go from a SK model with N spins, Σ_N, to one with $N+1$ spins, Σ_{N+1}. We shall make some physical assumptions on the organization of the configurations of Σ_N, inspired from recent results on the meaning of the RSB Ansatz of ref. [3]: the ultrametric organization of the states [6] and the independent exponential distribution of their free energies [7]. Assuming these properties for Σ_N, we shall show that they hold for Σ_{N+1}, and deduce all the other results of the replica treatment: value of the free energy, distribution of the local magnetizations in each state, and shape of the order parameter function.

For completeness, we first briefly review these physical properties.

The spin glass phase is charcterized by an infinite number of equilibrium states α with corresponding free energies F_α and local magnetizations m_i^α [4, 5]. A natural measure of the

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distance between two states α and β is their overlap

$$q^{\alpha\beta} = \frac{1}{N} \sum_i m_i^\alpha m_i^\beta ,$$

and the order parameter function is the probability distribution of these overlaps [4].

The space of equilibrium states is ultrametric [6]. This essentially means that the states are grouped into clusters: by choosing any scale of overlap q ($< q_M$), the space can be partitioned into nonoverlapping clusters such that two states in the same cluster have an overlap larger than q, while states in different clusters have an overlap smaller than q. The clusters at a scale $q' > q$ are subclusters of those at the scale q.

All the states have the same free energy per spin $F_\alpha/N = F_0$ to leading order in N, but the $O(1/N)$ corrections $f_\alpha$vary from state to state and determine their probabilities

$$P_\alpha = \exp[-\beta f_\alpha]\left[\sum_\gamma \exp[-\beta f_\gamma]\right]^{-1}$$

(where β is the inverse temperature). The number of states at fixed free energy f is

$$\mathcal{N}_\rho(f - f_s) \sim \exp[\rho(f - f_s)] , \tag{1}$$

where f_s is a free-energy scale. Taking into account the fact that the states are grouped into clusters at the scale q, one finds that the distribution of the states inside the same cluster I is still given by (1), but the free-energy scale f_s^{I} depends on the cluster. The distribution of these f_s^{I} is the same as (1) but with a parameter $\rho' < \rho$ ([1]). This structre is reminiscent of the generalized random energy model [9].

We can now proceed to analyse what happens when one adds a new spin σ_0 to the system Σ_N *of* N spins $\{\sigma_1, ..., \sigma_N\}$. For simplicity, we shall keep to the case where there is no external magnetic field. The spin σ_0 interacts with the N other ones through a set of couplings K_i which are independent random variables with $\overline{K_i} = 0$ and $\overline{K_i K_j} = 1/N\delta_{ij}$. As we do not change the coupling of Σ_N, they verify $\overline{J^2} = 1/N$ instead of $\overline{J^2} = 1/(N+1)$. The correct rescaling of $\overline{J^2}$ can be absorbed into a rescaling of the temperature, and the change of free energy ΔF we find is related to the free-energy density f and the energy density e through

$$\Delta F = e/2 + f. \tag{2}$$

Let us consider the first stage of symmetry breaking. We suppose that there exist M equilibrium states $\alpha = 1, ..., M$ of Σ_N, with

$$q_0 = \frac{1}{N} \sum_i m_i^\alpha m_i^\beta, \quad \alpha \neq \beta; \quad q_1 = \frac{1}{N} \sum_i (m_i^\alpha)^2, \tag{3}$$

and their free-energy distribution is the $\mathcal{N}_\rho(f)$ of (1). The local field on site 0 in the state α, $h^\alpha = \sum_i k_i m_i^\alpha$, is a random variable which depends on the sample and on the state. The

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([1]) The alert reader might be worried about the compatibility of these formulae with formula (14) of ref. [7]. He should notice that all fomulae in [7] can be multiplied by an arbitrary function of $v = M \exp[-\rho f_c]$ without affecting their validity: this is a global change of energy scale which does not affect the distribution of probabilities. Furthermore formula (14) was at fixed probability of the cluster, while in the present work we fix its energy scale f_s. At f_s fixed, the probability still fluctuates. Taking into account these fluctuations, a detailed computation shows that both formulations are valid.

probability that, choosing a sample, the h^α take values $h^1, \ldots, h^M$ is

$$P(h_1, \ldots, h_M) = \int \prod_i \left(\mathrm{d}K_i \sqrt{\frac{N}{2\pi}} \exp\left[-\frac{N}{2} K_i^2 \right] \right) \prod_{\alpha=1}^{M} \delta(h^\alpha - \sum_i K_i m_i^\alpha); \tag{4}$$

introducing integral representations of the δ-functions, one gets

$$P(h_1, \ldots, h_M) = \int \frac{\mathrm{d}H}{\sqrt{2\pi q_0}} \exp\left[-\frac{H^2}{2q_0} \right] \prod_{\alpha=1}^{M} \left(\frac{\exp[-(h^\alpha - M)^2/2(q_1 - q_0)]}{\sqrt{2\pi(q_1 - q_0)}} \right). \tag{5}$$

Thus there is a common piece $H = (1/M) \sum_\alpha h^\alpha$ which depends on the sample (its distribution is a Gaussian of width $\overline{H^2} = q_0$), around this the h^α are uncorrelated Gaussian variables of width $\overline{(h^\alpha - H)^2} = q_1 - q_0$.

Each state α of Σ_N generates a state of Σ_{N+1} where the magnetization on the new site is $m_0^\alpha = \mathrm{tgh}\,(\beta h^\alpha)$, and the corresponding change of free energy Δf_α is the sum of three pieces:

— the energy of the spin 0 in its local cavity field:

$$\Delta F_1 = -m_0^\alpha h^\alpha = -h^\alpha \,\mathrm{tgh}\,\beta h^\alpha, \tag{6}$$

— the entropy of the new spin:

$$\Delta F_2 = -\frac{1}{\beta} [\ln(2\cosh\beta h^\alpha) - \beta h^\alpha \,\mathrm{tgh}\,\beta h^\alpha], \tag{7}$$

— the change of free energy ΔF_3 due to the rearrangement of the N spins in the presence of spin 0. We shall prove, later on, that

$$\Delta F_3 = -\frac{\beta}{2}(1 - q_M), \tag{8}$$

where q_M is the self-overlap of a state.

For a fixed sample the new distribution of free energies is still the exponential (1), but its scale has been shifted by a factor $\varphi(H, q_1 - q_0, \rho) - (\beta/2)(1 - q_1)$, where

$$\varphi(H, q, \rho) = -\frac{1}{\rho} \ln \int \frac{\mathrm{d}h}{\sqrt{2\pi q}} \exp\left[-\frac{h^2}{2q} \right] [2\cosh\beta h]^{\rho/\beta}, \tag{9}$$

performing the quenched averaged over H gives the average change of free energy:

$$\overline{\Delta F} = \int \frac{\mathrm{d}H}{\sqrt{2\pi q_0}} \exp\left[-\frac{H^2}{2q_0} \right] \varphi(h, q_1 - q_0, \rho) - \frac{\beta}{2}(1 - q_1). \tag{10}$$

The distribution of local field in the states at a fixed new free energy, normalized and averaged over the samples, is

$$P(h) = \int \frac{\mathrm{d}H}{\sqrt{2\pi q_0}} \exp\left[-\frac{H^2}{2q_0} \right] \frac{\exp[-(h - H)^2/2(q_1 - q_0)][2\cosh\beta h]^{\rho/\beta}}{\varphi(H, q_1 - q_0, \rho)}. \tag{11}$$

Finally, in order for the overlaps not to change at order $1/N$ when one adds σ_0, one must impose the consistency equations

$$\begin{cases} q_1 = \displaystyle\int \mathrm{d}h\, P(h)[\mathrm{tgh}\,\beta h]^2\,, \\ q_1 = \displaystyle\int \frac{\mathrm{d}H}{\sqrt{2\pi q_0}} \exp\left[-\frac{H^2}{2q_0}\right] \cdot \\ \qquad \cdot \left[\dfrac{\left(\int \mathrm{d}h/\sqrt{2\pi(q_1-q_0)}\right) \exp\left[-(h-H)^2/2(q_1-q_0)\right][2\cosh\beta h]^{\rho/\beta}\,\mathrm{tgh}\,\beta h}{\varphi(H,\, q_1-q_0,\, \rho)}\right]^2. \end{cases} \tag{12}$$

Using the expression (2) of ΔF, one can check that (10)-(12) are exactly the results of the replica method with one level of RSB [3, 10, 11].

Let us sketch the essential points of the computation at the second level of symmetry breaking. The states are grouped into clusters. The energy scales of the clusters have the distribution $\mathcal{N}_{\rho_1}(f_\mathrm{s}^\mathrm{I}-f)$, and the distribution of the free energies of the states inside the same cluster I is $\mathcal{N}_{\rho_2}(f-f_\mathrm{s}^\mathrm{I})$. The self-overlap of a state is q_2, the overlap of two states in the same cluster is q_1, and the overlap of states in different clusters is q_0. The local fields h^α on the site 0 are found to depend on three pieces. A field H which depends only on the sample and is a Gaussian variable with zero average and variance q_0. A field H^I which depends on the cluster. For fixed H, the H^I are independent Gaussian variables with average H and variance q_1-q_0. A field h^α which depends on the state. For fixed H and H^I, the h^α of the various states which belong to the same cluster I are independent Gaussian variables with average H^I and variance q_2-q_1.

Adding the new site, one finds the same structure of independent free-energy scales, and independent free energies of the states inside the clusters. The free-energy scale of each cluster is changed as in (10) by a factor $\varphi(H^\mathrm{I},\, q_2-q_1,\, \rho_2)-(\beta/2)(1-q_2)$, and the experimental distribution of the free-energy scales is shifted of a factor (for fixed H)

$$\Delta f_\mathrm{s}(H) = -\frac{1}{\rho_1}\ln \int \frac{\mathrm{d}H^\mathrm{I}}{\sqrt{2\pi(q_1-q_0)}} \cdot \exp\left[-\frac{(H^\mathrm{I}-H)^2}{2(q_1-q_0)}\right] \exp\left[-\rho_1\left[\varphi(H^\mathrm{I},\, q_2-q_1,\, \rho_2)-\frac{\beta}{2}(1-q_2)\right]\right]. \tag{13}$$

Averaging finally over the sample-dependent common drift H gives

$$\overline{\Delta F} = \int \frac{\mathrm{d}H}{\sqrt{2\pi q_0}} \exp\left[-\frac{H^2}{2q_0}\right] \Delta f_\mathrm{s}(H)\,. \tag{14}$$

This is exactly the result for $\overline{\Delta F}$ in the replica method with two RBS [3, 10, 11]. One can check again that the distribution of the local field and the coherence equations determining the q's are the same as those obtained with replicas. This discussion can be extended in a straightforward way to an arbitrary number of cluster hierarchies.

We still have to demonstrate formula (8) for the change of free energy due to the rearrangement of the N spins. An instructive proof follows from the generalization of our arguments at the level of the configurations.

In a given state α, at a given temperature, the set of relevant configurations has the following properties: the number of configurations of energy $\bar{E}+E$ ($\bar{E}$ is supposed to be of order N and E of order 1) is

$$\mathcal{N}(\bar{E}+E) \sim \exp\left[s(\bar{E})+\beta E\right] \tag{15}$$

and the mutual overlap of these configurations is the self-overlap of the state, $q^{\alpha\alpha} = q_M$.

Let us consider a set of M ($\gg 1$) such configurations. Each configuration $\mathscr{C}$ creates a local field $h^{\mathscr{C}}$ on the site 0. Following the same argument as in (5), one finds that, for fixed $\bar{h} = (1/M) \sum_{\mathscr{C}} h^{\mathscr{C}}$, the $h^{\mathscr{C}}$ are independent Gaussian random variables of width $1 - q_M$ and average $\bar{h}$. One has

$$\bar{h} = \sum_i k_i \left[\frac{1}{M} \sum_{\mathscr{C}} \sigma_i^{\mathscr{C}} \right] \sum_i K_i m_i^{\alpha}, \tag{16}$$

hence $\bar{h}$ is the average magnetic field h^{α} on site 0 in state α.

Adding the new spin, each configuration gives rise to two configurations with $\sigma_0 = \pm 1$ and energies $E^{\mathscr{C}} \mp h^{\mathscr{C}}$. We can safely assume that the distribution of energies and magnetic field in Σ_N are uncorrelated (in the expressions of the magnetic field on site 0 and of the energies the values of the spins are multiplied, respectively, by K_i and J_{ij} which are statistically independent). Then the distribution of energies (15) in Σ_{N+1} is multiplied by a factor

$$C = \int \frac{\mathrm{d}h}{\sqrt{2\pi(1 - q_M)}} \exp\left[- \frac{(h - h^{\alpha})^2}{2(1 - q_M)} \right] [\exp[\beta h] + \exp[-\beta h]] . \tag{17}$$

This change of normalization of the exponential has a double origin: a shift of entropy ΔS (shift of the vertical scale), and a shift of $\bar{E}$ into $\bar{E} + \Delta E$, which shifts the horizontal scale. The net change is $C = \exp[\Delta S - \beta \Delta E]$, which proves that the change of free energy is

$$\Delta F = -\frac{1}{\beta} \ln C = -\frac{1}{\beta} \ln [2 \cosh (\beta h_{\alpha})] - \frac{\beta}{2}(1 - q_M) . \tag{18}$$

We have recovered all the results of the replica method on purely physical grounds. In fact we start from the same hypotheses as in the replica method (but instead of being hidden in the form of the RSB ansatz they are explicit), and obtain the same results. These results are thus clarified and can be written very easily. For instance the existence of a dip at the origin in the distribution of local fields at low temperature follows from the following fact: in Σ_N there is an exponentially large number of states at a distance h above the ground state, and hence a nonvanishing probability that one of them will be the ground state of Σ_{N+1}, with a large local field.

The resulting picture possesses remarkable properties. The distribution of relevant states inside a cluster is very similar to the distribution of relevant configurations inside one pure state (exponential increase of the number of states with the free energy, and uncorrelated local fields around the average one [11]).

REFERENCES

[1] S. F. EDWARDS and P. W. ANDERSON: *J. Phys. F*, 5, 965 (1975).

[2] D. SHERRINGTON and S. KIRKPATRICK: *Phys. Rev. Lett.*, 32, 1792 (1975); S. KIRKPATRICK and D. SHERRINGTON: *Phys. Rev. B*, 27, 4384 (1978).

[3] G. PARISI: *Phys. Rev. Lett.*, 43, 1754 (1979); *J. Phys. A*, 13, L155, 1101, 1887 (1980).

[4] G. PARISI: *Phys. Rev. Lett.*, 50, 1946 (1983).

[5] A. HOUGHTON, S. JAIN and A. P. YOUNG: *J. Phys. C*, 16 L375 (1983); C. DE DOMINICIS and A. P. YOUNG: *J. Phys. A*, 16, 2063 (1983).

[6] M. MÉZARD, G. PARISI, N. SOURLAS, G. TOULOUSE and M. VIRASORO: *Phys. Rev. Lett*, 52, 1156 (1884); *J. Phys. (Paris)*, 45, 843 (1984).
[7] M. MÉZARD, G. PARISI and M. A. VIRASORO: *J. Phys. (Paris) Lett.*, 46, L217 (1985).
[8] D. J. THOULESS, P. W. ANDERSON and R. G. PALMER: *Phil. Mag.*, 35, 593 (1977).
[9] B. DERRIDA: *J. Phys. (Paris) Lett.*, 46, L401 (1985).
[10] J. R. L. DE ALMEIDA and E. J. LAGE: *J. Phys. C*, 16, 939 (1982).
[11] M. MÉZARD and M. A. VIRASORO: to appear in *J. Phys. (Paris)*.