
LE JOURNAL DE PHYSIQUE-LETTRES

J. Physique Lett. **46** (1985) L-985 - L-989

1^{er} NOVEMBRE 1985, PAGE L-985

Classification

Physics Abstracts

64.70P — 05.40 — 75.40

Diffusion in an ultrametric space : a simple case

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(Reçu le 17 juillet 1985, accepté le 11 septembre 1985)

Résumé. — Nous introduisons et résolvons un modèle simple de relaxation hiérarchique : la diffusion par sauts dans un espace régulier ultramétrique.

Abstract. — We introduce and solve a simple model of hierarchical relaxation : the diffusion by hopping in a regular ultrametric space.

A general property of the time relaxation of amorphous systems (glasses, spin glasses...) is the existence of a wide spectrum of characteristic times which leads to very slow relaxation. This type of behaviour can be understood if one assumes that there are many relevant degrees of freedom which relax hierarchically, as it was shown in [1].

From the point of view of equilibrium statistical mechanics, one of the best understood among these systems is the mean field theory of spin glasses. This model is known to possess an infinite number of equilibrium states which form an ultrametric space [2]. Ultrametricity means precisely that there is a hierarchical organization : the states can be grouped into clusters, themselves regrouped into superclusters etc... It is also a natural property for the relaxational dynamics of a system by thermal activation over energy barriers (at low temperature) : if one defines the « distance » between two energy minima as the height of the lowest lying saddle point which allows to go from one minimum to the other, this distance can be shown to be ultrametric.

In this note we want to show on a very simple example how the diffusion by hopping in an ultrametric space gives rise to hierarchical relaxation. We consider a system which can be in

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any of N states $a = 1, 2, \dots, N$, and we suppose that the characteristic hopping times $1/w_{ab}$ between them ($w_{ab} = w_{ba}$ is the transition probability per unit time from a to b or from b to a) define an ultrametric distance : given three states a, b and c , one has :

$$1/w_{ab} \leq \sup (1/w_{ac} ; 1/w_{bc}) . \tag{1}$$

It is convenient to represent such an ultrametric space by a tree as in figure 1 in which the extremities of the branches represent the states and w_{ab} has a Parisi [3] structure and only depends on the index of the lowest node common to a and b . In the following we study in particular the model in which all the states have the same equilibrium probability (a system with non equal probabilities which are sample dependent random variables distributed as in the mean field theory of spin glasses has been studied in [4] and is only touched upon here). There are R levels of the hierarchy, at each level l the transition rate is w_l and the number of branching is α_l , which means (Fig. 1) that at that level there is a bifurcation with α_l clusters, each of which contains N_l states

$$N_l = \alpha_1 \dots \alpha_{l-1} . \tag{2}$$

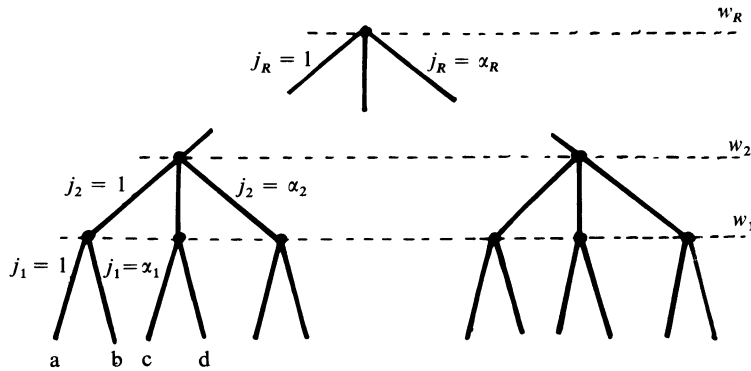


Fig. 1. — Ultrametric structure of the states. Here a, b, \dots denote various states of the system. A state is characterized by R numbers $j_1 \dots j_R$ with $j_l = 1, 2, \dots, \alpha_l$. The transition rate from one state to the other only depends on the closest common node (i.e. the number of successive distinct values $j_1 \neq j'_1, j_2 \neq j'_2 \dots$ for the two states) e.g. $w_{ab} = w_1, w_{bc} = w_2 \dots$

Note that for convenience we use here a labelling distinct (inverse) of the usual one. We now characterize the N states by R indices $(j_1 \dots j_R)$, $1 \leq j_l \leq \alpha_l$ and $N = \alpha_1 \dots \alpha_R$. The indices name the branches of the tree one must follow in order to go from the common original root (R th level) to one of the N end branches (1st level). The probability of being at time t in a state $a \equiv (j_1 \dots j_R)$, $P_a(t)$, satisfies the master equation

$$\dot{P}_a(t) = \sum_{a \neq b} [w_{ab} P_b(t) - w_{ba} P_a(t)] , \quad a = 1, 2, \dots, N . \tag{3}$$

We solve (3) iteratively, computing first the probability of occupation of the largest cluster, then increasing the resolution by therefrom computing the probability for smaller and smaller clusters.

At the largest scale, there are α_R clusters of states $j_R = (1, 2, \dots, \alpha_R)$. We denote by $P_{j_R}^{(R)}(t)$ the probability of being in one of these clusters

$$P_{j_R}^{(R)}(t) = \sum_{a \in j_R} P_a(t) = \sum_{j_1=1}^{\alpha_1} \dots \sum_{j_{R-1}=1}^{\alpha_{R-1}} P_{j_1 j_2 \dots j_R}(t). \tag{4}$$

Summing (3) over all states in cluster j_R we get

$$\dot{P}_{j_R}^{(R)}(t) = -Nw_R(P_{j_R}^{(R)}(t) - 1/\alpha_R) \tag{5}$$

i.e.

$$P_{j_R}^{(R)}(t) = 1/\alpha_R + e^{-t/t_R}(P_{j_R}^{(R)}(0) - 1/\alpha_R) \tag{6}$$

$$1/t_R = w_R N. \tag{7}$$

Summing now (3) over the subcluster $j_R j_{R-1}$ one obtains likewise $P_{j_{R-1} j_R}^{(R-1)}(t)$ until by iteration one derives the probability for the state $a \equiv j_1 \dots j_R$. We get

$$P_{j_1 \dots j_R}^{(1)}(t) = N_1 \left\{ \sum_{l=1}^R e^{-t/t_l} (P_{j_1 \dots j_R}^{(l)}(0)/N_l - P_{j_1 \dots j_R}^{(l+1)}(0)/N_{l+1}) + 1/N_{R+1} \right\} \tag{8}$$

with

$$1/t_l = \sum_{r=l}^R (w_r - w_{r+1}) N_{r+1} \tag{9}$$

$$N_l = \alpha_1 \dots \alpha_{l-1} \tag{10}$$

where $w_{R+1} = 0, N_1 = 1$. Equations (8) to (10) are valid for intermediate subclusters $P^{(l_0)}$ by replacing the index 1 by l_0 in (8). They extend to the case where states do not have the same probability but are weighted at equilibrium by a Boltzmann weight $\exp - \beta f_a$. In that case, taking into account that w_{ab} is changed into $w_{ab} \exp - \beta f_a$ (for detailed balance) it suffices, in equations (8) and (9) to make the replacement [4]

$$N_l \rightarrow Z_{j_1 \dots j_R}^{(l)} \equiv \sum_{j_1 \dots j_{l-1}} \exp - \beta f_{j_1 \dots j_R}. \tag{11}$$

Returning to the case where all states are equivalent, let us take the initial system in a given state $(j_1^0 \dots j_R^0) \equiv a_0$. The probability for return to that state (autocorrelation functions) is then

$$P_{j_1^0 \dots j_R^0}^{(1)}(t) = \sum_{l=1}^R (\alpha_1 \dots \alpha_{l-1})^{-1} (1 - 1/\alpha_l) e^{-t/t_l} + 1/N. \tag{12}$$

This is a superposition of exponentials typical of a hierarchical relaxation that exhibits distinct behaviours depending upon the choice of the sequences α_i, w_i . When the relaxation times t_i form a sequence where each time scale is much larger than the previous one and when the branchings α_i are large, one can write

$$1/t_l = \sum_{r=l}^R w_r N_{r+1} \simeq w_l N_{l+1} \tag{13}$$

$$P^{(1)}(t) = \sum_{l=1}^R e^{-t/t_l}/N_l + 1/N. \tag{14}$$

Following the argument of reference [5] one can look for the behaviour of $P^{(1)}(t)$ for

$$t_{l_0-1} \ll t < t_{l_0} \tag{15}$$

where $P^{(1)}(t)$ reduces to

$$P^{(1)}(t) \simeq \sum_{l=l_0(t)}^R 1/N_l \sim 1/N_{l_0}. \quad (16)$$

Here $l_0(t)$ is then obtained by

$$1/t_{l_0} \simeq 1/t \simeq w_{l_0} N_{l_0+1} \quad (17)$$

with both $1/N_l$ and $w_l N_{l+1}$ rapidly decreasing functions of l .

For $N_l \sim (\alpha)^{l-1}$ (i.e. $\alpha_l = \alpha$) and (i) $t_l \sim (w/\alpha)^l$ or (ii) $t_l \sim l^\gamma$, solving for $l_0(t)$ one obtains

$$(i) \quad P^{(1)}(t) \sim t^{-\ln \alpha / \ln(w/\alpha)} \quad (18)$$

$$(ii) \quad P^{(1)}(t) \sim \exp - (\ln \alpha) t^{1/\gamma}. \quad (19)$$

For $N_l \sim l^\nu$, we correspondingly get

$$(iii) \quad P^{(1)}(t) \sim (\ln(w/\alpha) / \ln t)^\nu \quad (20)$$

$$(iv) \quad P^{(1)}(t) \sim t^{-\nu/\gamma}. \quad (21)$$

These crude estimates can be refined even when the ratio $\lambda \equiv w/\alpha$ is not large. Indeed from (9), with again $\alpha_l = \alpha$, $w_l = w_0/w^l$ one has

$$1/\tau_l \equiv t_1/t_l = (\lambda^{-l+1} - \lambda^{-R}) / (1 - \lambda^{-R}). \quad (22)$$

Here we take t_1 , the shortest time scale as the time unit, $\hat{t} \equiv t/t_1$ and equation (22) now displays two distinct regimes with a critical value $\lambda_0 = 1$, viz. (i) if $\lambda < 1$, $\tau_l^{-1} = 1$ and one recovers exponential relaxation; (ii) if $\lambda > 1$, then

$$P^{(1)}(\hat{t}) = \sum_{l=1}^R \alpha^{-l+1} \exp - \hat{t} \lambda^{-l+1} + 1/N. \quad (23)$$

For large \hat{t} , $P^{(1)}(\hat{t})$ is easily calculated (e.g. transforming the sum over l into an integral and using a saddle point expansion). This again yields the same large time expression (18) for (i) now valid for all $\lambda \equiv w/\alpha > 1$. In contradistinction, because it is more difficult to define well separated time scales with a power law than with an exponential, the result is modified for (ii) and the improved calculation now replaces γ by $1 + \gamma$ in (19).

A comment is in order about the above power law (18). A similar formula has appeared in the work of Huberman and Kerszberg [5]. The model they have studied is the diffusion in a one dimensional box with hierarchical barriers (see also Ref. [6]). It should be emphasized that this model is different from ours: they still have the notion of an Euclidean distance since one can jump over a given barrier only when starting from a neighbouring cell. Hence their transition matrix is not fully ultrametric contrary to ours. The two models have the same asymptotic behaviour in the limiting case in which the decrease rate of the transition probability w is very large. In the one dimensional model of reference [5] one also expects that for $w \rightarrow 1$, the relaxation would be that of a random walk $P(t) \sim t^{-1/2}$ whereas in the same limit our model returns to normal exponential decay. It would be interesting to know whether in the Huberman-Kerszberg model there exists, like in the true ultrametric space, a critical value of the ratio w/α separating a region of normal decay $P(t) \sim t^{-1/2}$ from an anomalous one $P(t) \sim t^{-\delta}$, $\delta < 1/2$.

While we were writing up this work, we have received a preprint by Ogielsky and Stein [7] who have diagonalized the transition matrix in the case of an homogeneous ultrametric space with 2 branchings per level and obtained results similar to ours. We thank G. Toulouse for pointing out their work to us.

References

- [1] PALMER, R. G., STEIN, D. L., ABRAHAMS, E. and ANDERSON, P. W., *Phys. Rev. Lett.* **53** (1984) 958.
 - [2] MÉZARD, M., PARISI, G., SOURLAS, N., TOULOUSE, G. and VIRASORO, M., *Phys. Rev. Lett.* **52** (1984) 1156; *J. Physique* **45** (1984) 843.
 - [3] PARISI, G., *Phys. Rev. Lett.* **43** (1979) 1754.
 - [4] DE DOMINICIS, C., ORLAND, H. and LAINÉE, F., *J. Physique Lett.* **46** (1985) L-463.
DE DOMINICIS, C., ICM 1985, preprint.
 - [5] HUBERMAN, B. A. and KERSZBERG, M., *J. Phys. A* **18** (1985) L 331.
 - [6] GROßMANN, S., WEGNER, F. and HOFFMANN, K. H., *J. Physique Lett.* **46** (1985) L-575.
 - [7] OGIELSKI, A. T. and STEIN, D. L., 1985, preprint.
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