Diffusion in an ultrametric space: a simple case

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Résumé. — Nous introduisons et résolvons un modèle simple de relaxation hiérarchique : la diffusion par sauts dans un espace régulier ultramétrique.

Abstract. — We introduce and solve a simple model of hierarchical relaxation: the diffusion by hopping in a regular ultrametric space.

A general property of the time relaxation of amorphous systems (glasses, spin glasses...) is the existence of a wide spectrum of characteristic times which leads to very slow relaxation. This type of behaviour can be understood if one assumes that there are many relevant degrees of freedom which relax hierarchically, as it was shown in [1].

From the point of view of equilibrium statistical mechanics, one of the best understood among these systems is the mean field theory of spin glasses. This model is known to possess an infinite number of equilibrium states which form an ultrametric space [2]. Ultrametricity means precisely that there is a hierarchical organization: the states can be grouped into clusters, themselves regrouped into superclusters etc... It is also a natural property for the relaxational dynamics of a system by thermal activation over energy barriers (at low temperature): if one defines the «distance» between two energy minima as the height of the lowest lying saddle point which allows to go from one minimum to the other, this distance can be shown to be ultrametric.

In this note we want to show on a very simple example how the diffusion by hopping in an ultrametric space gives rise to hierarchical relaxation. We consider a system which can be in

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any of \( N \) states \( a = 1, 2, ..., N \), and we suppose that the characteristic hopping times \( 1/w_{ab} \) between them \( (w_{ab} = w_{ba} \) is the transition probability per unit time from \( a \) to \( b \) or from \( b \) to \( a \)) define an ultrametric distance: given three states \( a, b \) and \( c \), one has:

\[
1/w_{ab} \leq \sup \left( 1/w_{ac} ; 1/w_{bc} \right).
\]  

(1)

It is convenient to represent such an ultrametric space by a tree as in figure 1 in which the extremities of the branches represent the states and \( w_{ab} \) has a Parisi [3] structure and only depends on the index of the lowest node common to \( a \) and \( b \). In the following we study in particular the model in which all the states have the same equilibrium probability (a system with non equal probabilities which are sample dependent random variables distributed as in the mean field theory of spin glasses has been studied in [4] and is only touched upon here). There are \( R \) levels of the hierarchy, at each level \( l \) the transition rate is \( \omega_l \) and the number of branching is \( \alpha_l \), which means (Fig. 1) that at that level there is a bifurcation with \( \alpha_l \) clusters, each of which contains \( N_l \) states

\[
N_l = \alpha_1 \ldots \alpha_{l-1}.
\]  

(2)

Fig. 1. — Ultrametric structure of the states. Here \( a, b \ldots \) denote various states of the system. A state is characterized by \( R \) numbers \( j_1 \ldots j_R \) with \( j_i = 1, 2, ..., \alpha_i \). The transition rate from one state to the other only depends on the closest common node (i.e. the number of successive distinct values \( j_1 \neq j'_1, j_2 \neq j'_2 \ldots \) for the two states) e.g. \( w_{ab} = w_1, w_{bc} = w_2 \ldots \)

Note that for convenience we use here a labelling distinct (inverse) of the usual one. We now characterize the \( N \) states by \( R \) indices \( (j_1 \ldots j_R) \), \( 1 \leq j_i \leq \alpha_i \) and \( N = \alpha_1 \ldots \alpha_R \). The indices name the branches of the tree one must follow in order to go from the common original root (\( R \)th level) to one of the \( N \) end branches (1st level). The probability of being at time \( t \) in a state \( a = (j_1 \ldots j_R) \), \( P_a(t) \), satisfies the master equation

\[
\dot{P}_a(t) = \sum_{a \neq b} \left[ w_{ab} P_b(t) - w_{ba} P_a(t) \right], \quad a = 1, 2, ..., N.
\]  

(3)

We solve (3) iteratively, computing first the probability of occupation of the largest cluster, then increasing the resolution by therefrom computing the probability for smaller and smaller clusters.
At the largest scale, there are \( \alpha_R \) clusters of states \( j_R = (1, 2, ..., \alpha_R) \). We denote by \( P_{j_R}^{(R)}(t) \) the probability of being in one of these clusters

\[
P_{j_R}^{(R)}(t) = \sum_{a \in j_R} P_a(t) = \sum_{j_1=1}^{\alpha_1} \cdots \sum_{j_{R-1}=1}^{\alpha_{R-1}} P_{j_1j_2...j_R}(t).
\]  

(4)

Summing (3) over all states in cluster \( j_R \) we get

\[
\dot{P}_{j_R}^{(R)}(t) = -Nw_R(P_{j_R}^{(R)}(t) - 1/\alpha_R)
\]  

i.e.

\[
P_{j_R}^{(R)}(t) = 1/\alpha_R + e^{-t/\alpha_R} P_{j_R}^{(R)}(0) - 1/\alpha_R)
\]  

(5)

(6)

Summing now (3) over the subcluster \( j_{R-1}j_R \) one obtains likewise \( P_{j_{R-1}j_R}^{(R-1)}(t) \) until by iteration one derives the probability for the state \( a = j_1...j_R \). We get

\[
P_{j_1...j_R}^{(1)}(t) = N_1 \left\{ \sum_{l=1}^{R} e^{-t/\alpha_l}(P_{j_1...j_R}^{(l)}(0)/N_l - P_{j_1...j_R}^{(l+1)}(0)/N_{l+1}) + 1/N_{R+1} \right\}
\]  

(8)

with

\[
1/t_l = \sum_{r=1}^{R} (w_{r} - w_{r+1}) N_{r+1}
\]  

(9)

(10)

where \( w_{R+1} = 0, N_1 = 1 \). Equations (8) to (10) are valid for intermediate subclusters \( P_{j_1...j_R}^{(l_0)} \) by replacing the index 1 by \( l_0 \) in (8). They extend to the case where states do not have the same probability but are weighted at equilibrium by a Boltzmann weight \( \exp - \beta f_a \). In that case, taking into account that \( w_{ab} \) is changed into \( w_{ab} \exp - \beta f_a \) (for detailed balance) it suffices, in equations (8) and (9) to make the replacement [4]

\[
N_l \rightarrow Z_{j_1...j_R}^{(l)} \equiv \sum_{j_1...j_{l-1}} \exp - \beta f_{j_1...j_{l-1}}.
\]  

(11)

Returning to the case where all states are equivalent, let us take the initial system in a given state \( (j_1^0...j_R^0) \equiv a_0 \). The probability for return to that state (autocorrelation functions) is then

\[
P_{j_1^0...j_R^0}^{(1)}(t) = \sum_{l=1}^{R} (\alpha_1...\alpha_{l-1})^{-1} (1 - 1/\alpha_l) e^{-t/\alpha_l} + 1/N.
\]  

(12)

This is a superposition of exponentials typical of a hierarchical relaxation that exhibits distinct behaviours depending upon the choice of the sequences \( \alpha_l, w_l \). When the relaxation times \( t_l \) form a sequence where each time scale is much larger than the previous one and when the branchings \( \alpha_l \) are large, one can write

\[
1/t_l = \sum_{r=1}^{R} w_r N_{r+1} \approx w_l N_{l+1}
\]  

(13)

\[
P^{(1)}(t) = \sum_{l=1}^{R} e^{-t/\alpha_l}/N_l + 1/N.
\]  

(14)

Following the argument of reference [5] one can look for the behaviour of \( P^{(1)}(t) \) for

\[
t_{l_0-1} \ll t < t_{l_0}
\]  

(15)
where \( P^{(1)}(t) \) reduces to
\[
P^{(1)}(t) \simeq \sum_{l=1}^{R} \frac{1}{N_l} \sim \frac{1}{N_{l_0}}. \tag{16}
\]

Here \( l_0(t) \) is then obtained by
\[
\frac{1}{t_{l_0}} \simeq \frac{1}{t} \sim w_{l_0} N_{l_0+1}
\tag{17}
\]
with both \( 1/N_l \) and \( w_l N_{l+1} \) rapidly decreasing functions of \( l \).

For \( N_1 \sim (\alpha)^{l-1} \) (i.e. \( \alpha_l = \alpha \)) and (i) \( t_l \sim (w/\alpha)^l \) or (ii) \( t_l \sim l' \), solving for \( l_0(t) \) one obtains
\[
\begin{align*}
(i) & \quad P^{(1)}(t) \sim t^{-\ln w/\ln(w/\alpha)} \quad \tag{18} \\
(ii) & \quad P^{(1)}(t) \sim \exp - (\ln \alpha)^{1/\gamma}. \quad \tag{19}
\end{align*}
\]

For \( N_1 \sim l' \), we correspondingly get
\[
\begin{align*}
(iii) & \quad P^{(1)}(t) \sim (\ln (w/\alpha)/\ln t)^\gamma \quad \tag{20} \\
(iv) & \quad P^{(1)}(t) \sim t^{-\gamma}/. \quad \tag{21}
\end{align*}
\]

These crude estimates can be refined even when the ratio \( \lambda \equiv w/\alpha \) is not large. Indeed from (9), with again \( \alpha_l = \alpha, w_l = w_0/w' \) one has
\[
1/\tau_1 \equiv t_1/t_i = (\lambda^{-l+1} - \lambda^{-R})/(1 - \lambda^{-R}). \tag{22}
\]

Here we take \( t_1 \), the shortest time scale as the time unit, \( \hat{t} \equiv t/t_1 \) and equation (22) now displays two distinct regimes with a critical value \( \lambda_0 = 1 \), viz. (i) if \( \lambda < 1, \tau_i^{-1} = 1 \) and one recovers exponential relaxation; (ii) if \( \lambda > 1 \), then
\[
P^{(1)}(\hat{t}) = \sum_{l=1}^{R} \alpha^{-l+1} \exp - \hat{t}^\lambda - 1/N \tag{23}
\]

For large \( \hat{t} \), \( P^{(1)}(\hat{t}) \) is easily calculated (e.g. transforming the sum over \( l \) into an integral and using a saddle point expansion). This again yields the same large time expression (18) for (i) now valid for all \( \lambda \equiv w/\alpha > 1 \). In contradistinction, because it is more difficult to define well separated time scales with a power law than with an exponential, the result is modified for (ii) and the improved calculation now replaces \( \gamma \) by \( 1 + \gamma \) in (19).

A comment is in order about the above power law (18). A similar formula has appeared in the work of Huberman and Kerszberg [5]. The model they have studied is the diffusion in a one dimensional box with hierarchical barriers (see also Ref. [6]). It should be emphasized that this model is different from ours : they still have the notion of an Euclidean distance since one can jump over a given barrier only when starting from a neighbouring cell. Hence their transition matrix is not fully ultrametric contrary to ours. The two models have the same asymptotic behaviour in the limiting case in which the decrease rate of the transition probability \( w \) is very large. In the one dimensional model of reference [5] one also expects that for \( w \rightarrow 1 \), the relaxation would be that of a random walk \( P(t) \sim t^{-1/2} \) whereas in the same limit our model returns to normal exponential decay. It would be interesting to know whether in the Huberman-Kerszberg model there exists, like in the true ultrametric space, a critical value of the ratio \( w/\alpha \) separating a region of normal decay \( P(t) \sim t^{-1/2} \) from an anomalous one \( P(t) \sim t^{-\delta} \), \( \delta < 1/2 \).

While we were writing up this work, we have received a preprint by Ogielsky and Stein [7] who have diagonalized the transition matrix in the case of an homogeneous ultrametric space with 2 branchings per level and obtained results similar to ours. We thank G. Toulouse for pointing out their work to us.
References