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THE SIMPLEST SPIN GLASS

D J GROSS[1] and M MEZARD

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France*

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We study a system of Ising spins with quenched random infinite ranged p-spin interactions For $p \to \infty$, we can solve this model exactly either by a direct microcanonical argument, or through the introduction of replicas and Parisi's ultrametric ansatz for replica symmetry breaking, or by means of TAP mean field equations Although the model is extremely simple it retains the characteristic features of a spin glass We use it to confirm the methods that have been applied in more complicated situations and to explicitly exhibit the structure of the spin glass phase

1. Introduction

In recent years much effort has been devoted to the study of the low-temperature behaviour of systems of spins interacting via quenched random couplings – spin glasses [1] The characteristic feature of such a disordered system is the existence of many states of minimum free energy, separated by very high free energy barriers and unrelated by a symmetry one to another. As a consequence it is believed that in such systems ergodicity can break down, so that the equilibrium state will depend on the initial conditions.

Normally the first step towards understanding the phases of a given system is by means of mean field theory. In the case of spin glasses even mean field theory has proven to be very subtle. An appropriate infinite range spin glass model was proposed by Sherrington and Kirkpatrick (the SK model [2]) many years ago, but its solution has only been recently obtained By now there is general agreement that the SK model can be solved by means of the "replica method". This method is based initially on a mathematical trick which allows one, by introducing n replicas of the system and taking the $n \to 0$ limit, to replace quenched averages (which are hard) by annealed averages (which are easy) The basic observation, due to Parisi [3], is that the breaking of replica symmetry is physically related to the breakdown of ergodicity in the spin glass phase Parisi proposed a specific form for this replica symmetry breaking [6], which produces a stable mean field solution and which has

[1] On leave from Princeton University

* Laboratoire Propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris-Sud

a natural interpretation in terms of the structure of the space of free energy valleys [3–5]

These results have yielded a consistent picture of the mean field theory of a spin glass. However they rely heavily on a particular replica symmetry breaking scheme It is not a priori clear what physical principle is responsible for this very specific pattern, which possesses the very special ultra-metric property [5] The best evidence to date for the validity of Parisi's scheme is its stability [7] and the fact that it agrees with numerical experiments.

A few years ago it was pointed out by Derrida [8], that the SK model could be generalized to models involving p-spin interactions, and that these simplify in the limit of large p Derrida showed that the $p \to \infty$ SK model is equivalent to a random energy model, which consists of a collection of independent random energy levels He was then able to solve this model exactly, without recourse to replicas or other potentially dangerous tricks

In this paper we shall study the generalized p-spin SK model directly, with the aim of testing the methods that have been applied to the usual model and displaying in an explicit fashion the spin glass phase

Thus we shall apply the replica method to the p-spin model and analyse it within Parisi's hierarchical scheme When $p \to \infty$ it turns out that the first stage of replica symmetry breaking is exact Therefore we will obtain the analytic form of the order parameter function $q(x)$, and recover the values of the thermodynamic quantities (free energy, internal energy, magnetization) in agreement with the random energy model Furthermore we can analyse the structure of the space of free energy valleys in the spin glass phase, following [5], in terms of their statistical weights and the mutual overlap of their spins The physical interpretation of the order parameter function $q(x)$ in terms of the distribution of weights of the pure states of the system can be subjected to a critical test by evaluating the $1/N$ corrections to the entropy and comparing with Derrida's calculation within the random energy model

Another standard approach to the SK model is via the mean field equations of Thouless, Anderson and Palmer (the TAP equations) [9]. Again for $p \to \infty$ these simplify enormously. Since the system is totally frozen in the spin glass phase the cumbersome Onsager reaction terms can be neglected. We then can solve the model explicitly by calculating the density of TAP solutions and performing a canonical average over them, without the need to introduce replicas This approach reinforces the physical picture of the nature of the spin glass phase

We have attempted to write this paper so that it would be comprehensible to readers that are not spin glass experts, in the hope that the elucidation of the properties of this simplest of all spin glasses can serve as an introduction to the fascinating subject of the spin glass phase

The structure of the paper is as follows In sect 2, we review Derrida's demonstration of the equivalence of the $p \to \infty$ SK model with the random energy model and outline its solution Sect 3 is devoted to the replica method and its application to the $p \to \infty$ model. In sect. 4 we study the TAP equations for $p \to \infty$, and use the analysis of their solutions to gain further insight into the structure of the spin glass phase.

2. The random energy model

For the sake of completeness we shall review the argument of Derrida [8] on the equivalence of the p-spin SK model with the random energy model in the limit $p \to \infty$, as well as Derrida's solution of the latter

The generalized p-spin SK model describes a system of N Ising spins ($\sigma_i = \pm 1$) with infinite range p-spin quenched random interactions It is defined by the hamiltonian

$$\mathcal{H} = - \sum_{1 \leq i_1 < i_2 \dots < i_p \leq N} J_{i_1 i_2 \dots i_p} \sigma_{i_1} \sigma_{i_2} \cdot \dots \sigma_{i_p} \tag{1}$$

The interaction strengths are independent random variables which can be taken, for simplicity, to be gaussian. In order for the free energy to be extensive (i.e. proportional to N) the probability distribution of the J's must be scaled as follows.

$$P(J_{i_1 \dots i_p}) = \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left[-\frac{(J_{i_1 \dots i_p})^2 N^{p-1}}{J^2 p!} \right]. \tag{2}$$

For $p = 2$ this reduces to the standard SK model. We shall be interested in particular in the $p \to \infty$ limit of these models, where much simplification occurs Note that one must be careful to take the $p \to \infty$ limit *after* taking the thermodynamic limit, $N \to \infty$

Let $\{\sigma_i^{(1)}\}$ denote a given configuration of the spins with energy $\mathcal{H}(\sigma^{(1)})$ This energy depends, of course, on the particular choices of the couplings J The probability, $P(E)$, that it equals E is given by $P(E) = \overline{\delta(E - \mathcal{H}(\sigma^{(1)}))}$, where $\bar{O}(\langle O \rangle)$ stands for the average over the couplings (the thermodynamic average)

$$\begin{aligned} \overline{O(J, \sigma)} &= \int \prod \mathrm{d}J \, P(J) O(J, \sigma) \,, \\ \langle O(J, \sigma) \rangle &= \frac{1}{Z} \sum_{\sigma_i = \pm 1} \mathrm{e}^{-\beta \mathcal{H}(J, \sigma)} O(J, \sigma) \end{aligned} \tag{3}$$

Since the J have gaussian distribution, $P(E)$ is easily evaluated in the $N \to \infty$ limit to be

$$P(E) = \frac{1}{\sqrt{N \pi J^2}} \exp\left[-\frac{E^2}{J^2 N} \right] \tag{4}$$

Note that $P(E)$ is independent of p (which justifies the scaling of eq (2)) and of the spin configuration. This is a consequence of "gauge invariance", namely the fact that $\mathcal{H}(\sigma, J) = \mathcal{H}(\sigma', J')$ and $P(J) = P(J')$, where $J'_{i_1 \dots i_p} = J_{i_1 \dots i_p}(\sigma_{i_1} \sigma'_{i_1}) \dots (\sigma_{i_p} \sigma'_{i_p})$

Now consider two different spin configurations, $\{\sigma_i^{(1)}\}$ and $\{\sigma_i^{(2)}\}$ and calculate the probability, $P(E_1, E_2)$, that they have energies E_1 and E_2 respectively Due to the gauge invariance this can only depend on the *overlap*, q, between the two configurations

$$q^{(1,2)} \equiv \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(1)} \sigma_i^{(2)} \tag{5}$$

One finds (as $N \to \infty$)

$$\begin{aligned} P(E_1, E_2, q) &= \overline{\delta(E_1 - \mathscr{H}(\sigma^1))\delta(E_2 - \mathscr{H}(\sigma^2))} \\ &= [N\pi J^2(1+q^p)N\pi J^2(1-q^p)]^{-1/2} \\ &\times \exp\left[-\frac{(E_1+E_2)^2}{2N(1+q^p)J^2} - \frac{(E_1-E_2)^2}{2N(1-q^p)J^2}\right] \end{aligned} \tag{6}$$

The important point, discovered by Derrida [8], is that *if $\sigma^{(1)}$ and $\sigma^{(2)}$ are macroscopically distinguishable* ($|q^{(1,2)}| < 1$) *the energies are uncorrelated*, namely

$$P(E_1, E_2, q) \xrightarrow{p \to \infty} P(E_1) P(E_2) \quad (|q| < 1) \tag{7}$$

Of course when $q = 1$, $P(E_1, E_2, q) = P(E_1)\ \delta(E_1 - E_2)$

Similarly one can easily show that the probability distribution of n levels $\sigma^{(1)} \ldots \sigma^{(n)}$ with energies $E_1 \ldots E_n$, which can only depend on the overlaps $q^{(i,j)}$, factorizes when all $q^{(i,j)} < 1$

$$P(E_1, E_2 \ldots E_n; q^{(i,j)}) \xrightarrow{p \to \infty} \prod_{i=1}^{n} P(E_i) \quad (|q^{(i,j)}| < 1) \tag{8}$$

Therefore in the large $-p$ limit the energy levels become independent random variables The physics is identical to that of Derrida's random energy model, defined as a system of 2^N independent random energy levels distributed according to eq. (4)

Derrida has solved the random energy model, including the effect of an external magnetic field, as well as the leading $1/N$ corrections to the free energy For details see ref [8] Here we shall only briefly outline the microcanonical derivation of the free energy in zero field

Since the energy levels are independent random variables the average number of levels, $\langle n(E) \rangle$, of energy E is simply the total number of levels, 2^N, times the probability of finding E

$$\langle n(E) \rangle = \frac{1}{\sqrt{\pi N J^2}} \mathrm{e}^{N[\ln 2 - (E/NJ)^2]} \tag{9}$$

If $|E| < E_0 = N\sqrt{\ln 2}$ the average number of levels is very large Since the levels are statistically independent, the fluctuations are of order $1/\sqrt{\langle n(E) \rangle}$ and therefore negligible Thus $n(E) \sim \langle n(E) \rangle$ for $|E| < E_0$ On the other hand if $|E| > E_0$ there are simply no levels (with probability one) Therefore the entropy is

$$S(E)=N\left[\ln 2-\left(\frac{E}{NJ}\right)^2\right], \quad |E|<E_0 \tag{10}$$

Using $dS/dE=1/T$ one finds that the free energy is

$$\frac{F}{N}=\begin{cases}-T\ln 2-J^2/4T, & T>T_c\\ -\sqrt{\ln 2}, & T<T_c\end{cases} \tag{11}$$

The critical temperature, T_c, is

$$T_c=1/(2\sqrt{\ln 2}) \tag{12}$$

Below T_c the system gets stuck in the lowest available energy level, $E=-E_0$ and the entropy vanishes. Having completely disposed with the spin configurations, it is not easily seen that this model describes a spin glass Some evidence is provided by the behaviour of the magnetic susceptibility below T_c, which can be derived by similar arguments [8]. In the following we shall solve the $p\to\infty$ SK model directly and the spin glass nature of the low-temperature phase will be more apparent

3. Replica symmetry breaking

In this section we shall treat the p-spin generalized SK model defined by eq (1) (including a magnetic field) directly, and obtain the solution for $p\to\infty$ by the replica method. This model is a nice generalization of the standard SK ($p=2$) model, which shares with it all the essential features which are believed to be responsible for the unusual properties of spin glasses – quenched disorder and frustration* One expects that the low-temperature phase is a spin glass The characteristic feature of a spin glass phase is the existence of very many (infinite in the thermodynamic limit) states of minimum free energy (free energy valleys), which are unrelated one to another by any symmetry of the system, and which are separated by very high free energy barriers In the infinite range model, these barriers are infinitely high and are responsible for the breakdown of ergodicity Thus the particular valley into which the system will dynamically relax depends sensitively on the initial conditions.

Recently it has been realized that the best way of characterizing the spin glass phase is in terms of the space of equilibrium states (free energy valleys) of the system [3, 4] Each valley α can be assigned a statistical weight, P_α, determined by its free energy, F_α

$$P_\alpha=\frac{e^{-F_\alpha/T}}{\sum_\gamma e^{-F_\gamma/T}}\,. \tag{13}$$

* Note that for odd p the model loses the "time-reversal invariance" which holds for even p, i e symmetry under reversal of all the spins This is of some interest since it yields a situation where the spin glass transition is purer, namely it does not mix with a ferromagnetic transition for zero field

Because of the breakdown of ergodicity the mean value of any observable O is given by

$$\langle O\rangle = \sum_{\alpha} P_\alpha \langle O\rangle_\alpha , \qquad (14)$$

where $\langle O\rangle_\alpha$ is the mean value of O in the valley α. The valleys thus correspond to *pure states* of the system. In contrast to more conventional systems, one expects that there exists an infinite number of such states, unrelated one to another by any symmetry. Furthermore there does not exist any macroscopic way to turn an external field (as one does, say, in a ferromagnetic by applying a magnetic field) in order to pick out a particular pure state The system is necessarily described by the above *mixture* of pure states

A measure of the distance in the space of valleys is introduced naturally in the following way: let $m_i^\alpha = \langle\sigma_i\rangle_\alpha$ be the magnetization of the spin i in the valley α. The overlap, $q^{\alpha\beta}$, between two valleys is defined to be

$$q^{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} m_i^\alpha m_i^\beta \qquad (15)$$

To describe the structure of this space it is natural to define the probability, $P(q)$, that two valleys, picked at random, have overlap q:

$$P(q) \equiv \sum_{\alpha,\beta} P_\alpha P_\beta \delta(q - q^{\alpha\beta}) , \qquad (16)$$

and to characterize the structure of the spin glass by the average of $P(q)$ over the random couplings, $\overline{P(q)}$ In an ordinary Ising model there exists one pure state at high temperature (with $\langle m_i\rangle = 0$) and two pure states at low temperature (with $\langle m_i\rangle^{\pm} \gtrless 0$) with equal probability. Thus for high T we would have $P(q) = \delta(q)$ and for low T, $P(q) = \frac{1}{2}[\delta(q+m^2) + \delta(q-m^2)]$. In the case of a spin glass, however, there are an *infinite number* of pure states – and therefore one expects that q will take many values.

The standard method for performing averages over the quenched couplings is to introduce n replicas of the system, calculate annealed averages and take the $n\to 0$ limit [10]. Thus the average free energy can be obtained as

$$\overline{\ln Z} = \lim_{n\to 0} \frac{1}{n}(\overline{Z^n} - 1) , \qquad (17)$$

and $\overline{Z^n}$ can be calculated by introducing n replicas of the system, σ_i^a, $a = 1 \quad n$ In an Ising-like model (with a symmetric distribution of couplings), once the average over the couplings is performed the effective hamiltonian can only depend on the overlap function of the replicas, Q_{ab}

$$Q_{ab}(\sigma) \equiv \frac{1}{N}\sum_{i=1}^{N} \sigma_i^a \sigma_i^b . \qquad (18)$$

One can relate Q_{ab} to the order parameters, q, that describe the structure of the space of valleys by evaluating the average of $P(q)$ using replicas. One obtains [3]

$$\int \overline{P(q)}\, e^{uq}\, dq = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{a\neq b} e^{u\langle Q_{ab}\rangle}, \tag{19}$$

where $\langle Q_{ab}\rangle$ is the mean value of the replica overlap matrix

The effective hamiltonian of Q_{ab} is, of course, symmetric under a permutation of the replica indices. Thus one might have expected that $\langle Q_{ab}\rangle$ would be replica-symmetric, i e $\langle Q_{ab}\rangle = Q$ $(a\neq b)$. However, this means that $\bar{P}(q)=\delta(q-Q)$ and therefore there is only a single pure state, with self-overlap (the Edwards–Anderson order parameter) equal to Q In a true spin glass phase, as for example in the $p=2$ SK model, q ranges over a continuous spectrum Therefore $\langle Q_{ab}\rangle$ must be characterized by an infinite number of parameters. Consequently the replica symmetry must be drastically broken

In the $p\to\infty$ model we shall be able to calculate explicitly the function $P(q)$ (this cannot be done in the finite-p case) using the replica method to calculate $\overline{Z^n}$ (we hereafter set $J=1$)

$$\overline{Z^n} = \int \prod dJ_{i_1 \cdots i_p} P(J_{i_1 \cdots i_p}) \times \mathrm{Tr}_{(\sigma_i^a)}\left[\exp \beta \sum_{a=1}^{n}\left[\sum_{i_1< \cdots <i_p} J_{i_1 \cdots i_p}\sigma_{i_1}^a \cdots \sigma_{i_p}^a + h\sum_i \sigma_i^a\right]\right] \tag{20}$$

One easily obtains

$$\overline{Z^n} = \mathrm{Tr}_{(\sigma_i^a)} \exp\left[\tfrac{1}{4}\beta^2 N\left(n+\sum_{a\neq b} Q_{ab}^p(\sigma)\right)+\beta h\sum_{i,a}\sigma_i^a\right] \tag{21}$$

The spin trace can be performed by constraining $Q_{ab}(\sigma)$ to equal Q_{ab}, with the aid of a Lagrange multiplier matrix λ_{ab}. One then gets

$$\overline{Z^n} = e^{nN\beta^2/4}\int_{-\infty}^{+\infty}\prod_{a<b} d\,Q_{ab}\int_{-i\infty}^{+i\infty}\prod_{a<b}\frac{d\lambda_{ab}}{2\pi}\, e^{-NG(Q_{ab},\lambda_{ab})}, \tag{22}$$

$$G(Q_{ab},\lambda_{ab}) = -\frac{1}{4}\beta^2\sum_{a\neq b}Q_{ab}^p + \frac{1}{2}\sum_{a\neq b}\lambda_{ab}Q_{ab} - \ln \mathrm{Tr}_{(\sigma_a)}\exp\left[\frac{1}{2}\sum_{a\neq b}\lambda_{ab}\sigma_a\sigma_b + \beta h\sum_a \sigma_a\right] \tag{23}$$

Unlike the case $p=2$, the effective hamiltonian is not quadratic in Q_{ab}, which therefore cannot be eliminated. In the limit $N\to\infty$, $\overline{Z^n}$ is given by the dominant saddle-point of G, namely mean field theory is exact, and the average free energy is $+\beta\bar{F}/N = \lim_{n\to 0}[G/n - \frac{1}{4}\beta^2]$ Actually one must find the absolute *maximum* of G, not the minimum. This reversal is one of the strange features of the $n\to 0$ limit

Since the matrix of fluctuations (of Q_{ab} or λ_{ab}) has $\frac{1}{2}n(n-1)$ parameters, it acts, for $n<1$, on a space of negative dimensions. In this situation the role of negative and positive eigenvalues is switched [6] and stability requires that G be maximized!

In order to evaluate G explicitly one must impose some ansatz on the structure of Q_{ab}, and a corresponding structure on λ_{ab}. For example in the high-temperature phase, the replica-symmetric ansatz is reasonable since we expect only one pure state

$$Q_{ab}=Q\,,\qquad \lambda_{ab}=\lambda\,,\quad a\neq b. \tag{24}$$

In that case one gets

$$\frac{1}{n}\,G(Q,\lambda)\overset{n\to 0}{=}\tfrac{1}{4}\beta^2Q^p-\tfrac{1}{2}\lambda Q-\int_{-\infty}^{+\infty}\mathrm{D}z\,\ln\left[2\,\mathrm{ch}\,(z\sqrt{\lambda}+\beta h)\right], \tag{25}$$

where

$$\mathrm{D}z\equiv\frac{\mathrm{d}z}{\sqrt{2\pi}}\,\mathrm{e}^{-z^2/2}$$

The saddle-point equations are

$$\tfrac{1}{2}\beta^2pQ^{p-1}=\lambda\,,\qquad Q=\int \mathrm{D}z\,\mathrm{th}^2\,(z\sqrt{\lambda}+\beta h) \tag{26}$$

When $p=\infty$ there exists a unique saddle-point for *all* β, h

$$Q=\mathrm{th}^2\,(\beta h),\quad \lambda=0 \tag{27}$$

The resulting free energy is then calculated from (22) and (25), to be

$$\frac{\overline{F}}{N}=-\frac{1}{4T}-T\ln 2-T\ln \mathrm{ch}\,\frac{h}{T} \tag{28a}$$

This replica-symmetric solution is indeed stable for large T (we shall derive the precise phase diagram below) and reproduces correctly the value of the thermodynamic quantities in the high-temperature phase of the random energy model [8] This phase contains a single pure state $\bar{P}(q)=\delta(q-\mathrm{th}^2\,(\beta h))$, whose self-overlap is the square of the magnetization.

The entropy in this phase

$$S=\ln 2-\frac{1}{4T^2}+\ln \mathrm{ch}\,\frac{h}{T}-\frac{h}{T}\,\mathrm{th}\,\frac{h}{T}\,, \tag{28b}$$

clearly becomes negative for $T\leqslant T_1(h)$, and therefore there must be a phase transition at some $T_c\geqslant T_1(h)$. In fact, as is evident from the random energy model, we shall see that $T_c=T_1(h)$

Unlike the case in the $p=2$ model, the $q=\mathrm{th}^2(\beta h)$ solution is the only replica-symmetric one at all temperature (for $p\to\infty$) In fact, an analysis of the stability of this solution within the complete replica space (à la De Almeida–Thouless [11]) shows that it is always locally stable in zero field, as soon as $p>2$ In this respect, the $p=2$ SK model is somewhat special The spin glass transition must be (for $p>2$) a first-order one, at least as far as the order parameter function $q(x)$ is concerned In fact we shall show, in the $p\to\infty$ case, that the Edwards–Anderson order parameter, $q(1)$, jumps from 0 to 1 at T_c However, since the order parameter is a function, and the discontinuity appears only on a set of zero measures, the transition turns out to be of second order in the thermodynamic sense.

In order to obtain the low-temperature spin glass phase, we must break replica symmetry, allowing Q_{ab} to depend, in general, on an infinite number of parameters The most general form of such a Q_{ab} is not known. Parisi has given a particular ansatz, which describes a hierarchical breaking of replica symmetry [6] For $p=2$ this does yield a stable maximum of F and agrees with numerical results For $p=\infty$ we shall show that it leads to the correct solution. (Note that the equations for a saddle-point of $G(Q,\lambda)$ will force λ_{ab} to have the same structure as Q_{ab})

Parisi's ansatz for Q_{ab} can be described by means of the following recursive algorithm

(i) First breaking the n replicas are grouped in n/m_1 clusters of m_1 replicas. Any two replicas, $a\neq b$, within the same cluster have overlap $Q_{ab}=q_1$, whereas replicas in different clusters have overlap $Q_{ab}=q_0\leqslant q_1$

(ii) Second breaking each cluster of size m_1 is broken up into m_1/m_2 sub-clusters of m_2 spins Any two replicas, $a\neq b$, in a sub-cluster have overlap $q_2\geqslant q_1$, the other overlaps remain unchanged

One continues to iterate this procedure, thus obtaining the general k-breaking situation, defined by

$$n\geqslant m_1\geqslant m_2\cdots\geqslant m_k\geqslant 1\,,$$
$$q_k\geqslant q_{k-1}\geqslant\cdots\geqslant q_1\geqslant q_0 \tag{29}$$

(Note that to achieve the continuation to $n=0$, one must let the m_i be continuous and reverse the inequalities in (29), i e for $n=0$ $0\leqslant m_1\leqslant\cdots\leqslant m_k\leqslant 1$)

The matrix obtained in the kth step by this procedure is best described by a genealogical tree with k generations, as shown in fig. 1 It can be parametrized by the function $x(q)$ – which equals the fraction of pairs of replicas with overlap $Q_{ab}\leqslant q$ The defining characteristic of Parisi's scheme of replica symmetry breaking is its ultrametric structure It is clear from the tree that if we consider three distinct replicas a, b, c, then the smallest two of the overlaps Q_{ab}, Q_{bc} and Q_{ac} must be equal

In the limit of infinite K, q will be continuous, and we can define $q(x)$ to be the inverse of $x(q)$ The physical meaning of $q(x)$ is evident from (19)

$$\int \overline{P(q)}\,\mathrm{e}^{uq}\,\mathrm{d}q=\int_0^1 \mathrm{d}x\,\mathrm{e}^{uq(x)}\,,$$
$$\overline{P(q)}=\frac{\mathrm{d}x}{\mathrm{d}q} \tag{30}$$

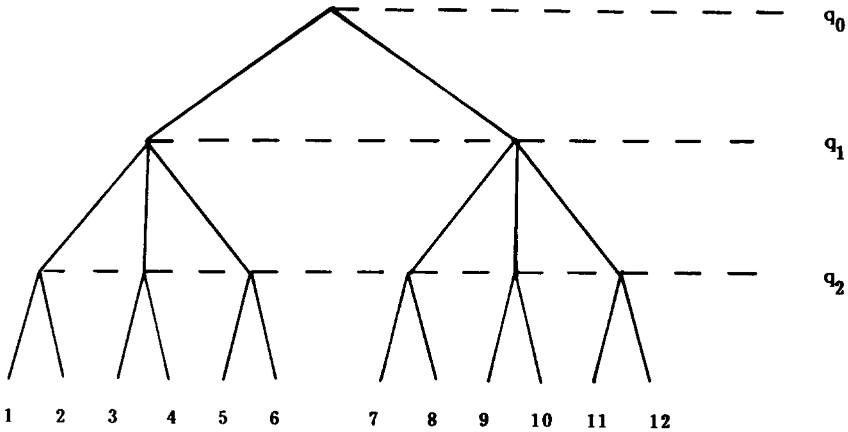

Fig 1 Parisi's ultrametric ansatz for replica symmetry breaking, described here for $n=12$, $m_1=6$, $m_2=2$ Replica indices $a, b = 1, \quad , 12$, are the extremities of the branches The value of the matrix element Q_{ab} is q_0, q_1 or q_2, depending on the closest common ancestor of a and b (For instance $Q_{34}=q_2$, $Q_{57}=q_0$)

Thus $x(q)$, the fraction of pairs of replicas with overlap $\leq q$, equals $\int_0^q \bar{P}(q')\,dq'$, the average fraction of pairs of valleys with overlaps $\leq q$

Let us now return to the p-spin SK model and consider the first step ($k=1$) in Parisi's scheme G is then a function of q_0, q_1, λ_0, λ_1 and $m=m_1$, and is given by

$$\frac{1}{n}G = \ln 2 - \tfrac{1}{4}\beta^2(mq_0^p + (1-m)q_1^p) + \tfrac{1}{2}(m\lambda_0 q_0 + (1-m)\lambda_1 q_1)$$

$$-\tfrac{1}{2}\lambda_1 + \frac{1}{m}\int Dz_0 \ln \int Dz_1 \,\mathrm{ch}^m(z_0\sqrt{\lambda_0} + z_1\sqrt{\lambda_1-\lambda_0} + \beta h) \tag{31}$$

For $p\to\infty$ the saddle-point equations are easy to solve First $\partial G/\partial q_i = 0$ implies

$$\lambda_i = \tfrac{1}{2}\beta^2 p q_i^{p-1} \tag{32}$$

For non-trivial symmetry breaking we must have $q_0 < q_1 \leq 1$, thus $\lambda_0 = 0$ If q_1 is also <1 then $\lambda_1 = 0$, in which case we will recover the symmetric solution $q_0 = q_1 = \mathrm{th}^2(\beta h)$. Hence $q_1 = 1$ and $\lambda_1 \sim \infty$

In this circumstance the double integral in G is easily calculated, and we obtain ($\lambda_1 \sim \infty$, $\lambda_0 \sim 0$)

$$\frac{1}{n}G = -\tfrac{1}{4}\beta^2(mq_0^p + (1-m)q_1^p) + \tfrac{1}{2}(m\lambda_0 q_0 + (1-m)\lambda_1 q_1) - \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}m\lambda_1$$

$$+\frac{1}{m}\ln(2\,\mathrm{ch}\,(m\beta h)) - \tfrac{1}{2}m\lambda_0\,\mathrm{th}^2(m\beta h) + \mathrm{O}(\lambda_0^2, 1/\lambda_1) \tag{33}$$

Differentiating with respect to λ_i then yields

$$q_0 = \mathrm{th}^2(\beta m h)\,, \qquad q_1 = 1\,, \tag{34}$$

consistent with our assumption Finally the variation with respect to m gives

$$m^2\beta^2 = 4\,[\ln 2 + \ln \operatorname{ch}(m\beta h) - m\beta h \operatorname{th}(m\beta h)] \tag{35}$$

This equation tells us that $m\beta = \beta_c$ is independent of the temperature, and β_c is given by

$$\beta_c^2 = 4[\ln(2\operatorname{ch}(\beta_c h)) - \beta_c h \operatorname{th}(\beta_c h)] \tag{36}$$

Since $m \leq 1$, the solution exists only for $T < T_c = 1/\beta_c$ (if m were greater than one, we would obtain negative weights in eq (19), in contradiction with the interpretation of $P(q)$ as a probability density)

T_c is precisely the value of the temperature $T_1(h)$, at which the entropy, eq (28b), of the high-temperature solution turns negative, and coincides with the critical line of the random energy model The free energy obtained for $T < T_c(H)$ can easily be calculated, using the above solution.

$$\frac{\overline{F}}{N} = -\frac{1}{2T_c} - h \operatorname{th}\frac{h}{T_c}, \tag{37}$$

precisely the result found by Derrida [8], for the low-temperature phase of the random energy model The magnetization is given by $m = \operatorname{th}(h/T_c)$ and the magnetic susceptibility is temperature-independent ($\chi = 1/T_c \operatorname{ch}^2(h/T_c)$), as is also true in the SK model [6].

In the $p = 2$ SK model one must go to the $k = \infty$ level of replica symmetry breaking Here *the first breaking of replica symmetry gives the exact answer.* Indeed we prove, in appendix A, that for the general kth-order breaking the only saddle-point is the one derived above. This phase is thus characterized by only 2 values of q $q(x) = \operatorname{th}^2(\beta_c h)\theta(T/T_c - x) + \theta(x - T/T_c)$, and $\overline{P(q)} = (T/T_c)\,\delta(q - \operatorname{th}^2(\beta_c h)) + (1 - T/T_c)\delta(q-1)$

The peak at $q = 1$ means that the self-overlap in a given valley, i.e the local magnetization, is maximal ($m_i = \pm 1$) Thus in the low-temperature phase the system is completely frozen, within each pure state there are no fluctuations of the magnetization. The peak at $q = \operatorname{th}^2(\beta_c h)$ means that two different valleys have an overlap equal to the square of the magnetization, i e the valleys are as far apart from one another as they could possibly be

It might seem that there are only two valleys, however this is not the case Following [5] we can calculate the distribution of the weights, P_α, of different clusters Choose an overlap scale, q, and group together all valleys with overlap larger than q, into clusters labelled by I, with weights $P_I = \sum_{\alpha \in I} P_\alpha$ It then follows that the average number of clusters of a given weight P_I is given by

$$f(P) = \overline{\sum_I \delta(P_I - P)} = \frac{P^{y-2}(1-P)^{-y}}{\Gamma(y)\Gamma(1-y)}, \tag{38}$$

which is a function of $y(q) = \int_q^1 \overline{P(q')}\, dq'$, the probability that the overlap is greater than q

If we choose $\mathrm{th}^2(\beta_c h) < q < 1$, then $y(q) = 1 - T/T_c$ and each cluster contains precisely one pure state Therefore the average number of pure states with weight $P_\alpha = P$ equals

$$f(P) = \overline{\sum_\alpha \delta(P_\alpha - P)} = \frac{P^{-1-T/T_c}(1-P)^{-1+T/T_c}}{\Gamma(T/T_c)\Gamma(1-T/T_c)} \tag{39}$$

This allows us to calculate the total number of pure states, which equals $N_v = \sum_\alpha 1 = \int_0^1 \mathrm{d}P f(P) = \infty$ The divergence occurs because of the existence of many valleys with small weight ($P \sim 0$). If we introduce a cutoff, $P \geqslant \varepsilon$, for the valley weights, $N_v(\varepsilon)$ blows up as

$$N_v(\varepsilon) \sim (1/\varepsilon)^{T/T_c}. \tag{40}$$

The fact that N_v increases with increasing temperature is somewhat surprising In general one might expect that as the temperature is lowered, the mean free energy is also lowered and more free energy valleys are explored. In our case, however, the system is frozen for $T \leqslant T_c$, F remains constant for $T \leqslant T_c$, and the only things that change are weights of the different valleys Valleys with large energies become less significant as T is lowered, and this leads to the decrease of N_v with temperature This result will be confirmed and explained further in sect 4

Even though there are an infinite number of pure states, most have very small weight and are insignificant. In fact the mean weight is given by

$$\overline{\sum_\alpha P_\alpha^2} = 1 - T/T_c. \tag{41}$$

To test the correctness of the above interpretation of the structure of the pure states in the spin glass phase, we can calculate the $1/N$ corrections to the theory and compare with Derrida's calculations within the random energy model [8]. We have seen that since the free energy is frozen for $T \leqslant T_c$, the $0(N)$ contribution to the entropy vanishes: $S/N \xrightarrow{N\to\infty} 0$. Now, if the system is in a mixture of pure states, α, the entropy is given by

$$S = \sum_\alpha P_\alpha S_\alpha - \sum_\alpha P_\alpha \ln P_\alpha, \tag{42}$$

where $S_\alpha = -\partial F_\alpha/\partial T$ is the entropy within the valley α, and $I = -\sum_\alpha P_\alpha \ln P_\alpha$ is the mutual entropy of the valleys, sometimes called the complexity [12]. Since the system is completely frozen within each valley ($m_i = \pm 1$), it is reasonable to assume that each $S_\alpha = 0$ (we cannot rigorously prove this to order $1/N$) Thus the entropy equals the complexity

$$\bar{S} = \bar{I} = -\overline{\sum_\alpha P_\alpha \ln P_\alpha} = -\int_0^1 P \ln P f(P)\,\mathrm{d}P$$
$$= \Gamma'(1) - \frac{\Gamma'(1-T/T_c)}{\Gamma(1-T/T_c)} \tag{43}$$

The specific heat per spin is then equal to

$$C=\frac{1}{N}\frac{T}{T_c}\left[\frac{\Gamma''(1-T/T_c)}{\Gamma'(1-T/T_c)}-\left(\frac{\Gamma'(1-T/T_c)}{\Gamma(1-T/T_c)}\right)^2\right]. \tag{44}$$

These $1/N$ corrections to F are in complete agreement with the results of [8], and confirm the physical interpretation of replica symmetry breaking developed in [3, 5]

4. The TAP equations

In this section we shall probe the structure of the $p=\infty$ SK model from the point of view of another time-honoured approach to spin glasses – that of the mean field equations of Thouless, Anderson and Palmer (TAP) [9] Our purpose is not to solve the model for a third time, but rather to gain further insight into the structure of the spin glass phase.

The TAP equations are mean field equations for a particular realization of the hamiltonian (1), which determine the local magnetization $m_i=\langle\sigma_i\rangle_J$ These differ from the naive mean field equations:

$$\mathrm{th}^{-1} m_i=\frac{1}{T}\sum_{i_2<\ldots<i_p} J_{i i_2 \ldots i_p} m_{i_2}\ldots m_{i_p}+\frac{h}{T}, \tag{45}$$

by the addition of Onsager-like reaction terms [9] The modification amounts to subtracting for each m_{i_j} in (45), the part of magnetization due to m_i However, this is proportional to the susceptibility $\chi_{i i_j}=(1-m_{i_j}^2)/T$, and we already know that in the $p=\infty$ model the system is frozen for $T\leqslant T_c$ and all $m_i=\pm 1$ Therefore we shall simply ignore these corrections. One could, presumably, prove directly from the full TAP equations that m_i had to equal ± 1 for $p=\infty$; we shall simply take this result from the previous solution of the model*.

The TAP equations will then be satisfied in the following fashion the sum on the right-hand side of (45) will diverge (as $p\to\infty$), as we shall see below, as $\sqrt{p}$, and therefore we will get a solution as long as

$$m_i=\mathrm{sgn}\left(\sum_{i_2<\ldots<i_p} J_{i i_2 \ldots i_p} m_{i_2}\ldots m_{i_p}\right) \tag{46}$$

This equation is actually valid for any p in the $T\to 0$ limit (for $h=0$), where the m_i are frozen to be ± 1 In our case it holds for all h and $T\leqslant T_c$ The free energy of a given solution (again for $p=\infty$, where $m_i=\pm 1$) is

$$F\{m_i\}=-\sum_{i_1<i_2<\ldots<i_p} J_{i_1 i_2 \ldots i_p} m_{i_1}\ldots m_{i_p}-\sum_i h m_i \tag{47}$$

Due to the extreme simplicity of these equations we shall be able to compute the

* We thank C de Dominicis for interesting discussions on the generalization of TAP equations to the p-spin SK model

number of solutions and the free energy exactly, using arguments similar to those used in the microcanonical solution of the random energy model given in sect. 2

Let us first compute the number of solutions of a given energy $N(E)$, for zero field We choose a particular configuration $\{m_i\}$ among the 2^N possibilities ($m_i = \pm 1$, $i = 1, \ldots, N$) Then we calculate the probability $P(E, \{m_i\})$ that, when averaged over all choices of couplings $J_{i_1 \ldots i_p}$, this configuration solves (46) and has free energy $F(\{m_i\}) = -E$ Then

$$N(E) = \sum_{\{m_i\}} P(E, \{m_i\}) \tag{48}$$

Since the distribution of the J's, eq. (2), is invariant under the "gauge transformation"

$$J_{i_1 \ldots i_p} \to \hat{J}_{i_1 \ldots i_p} = J_{i_1 \ldots i_p} m_{i_1} \ldots m_{i_p}, \tag{49}$$

the probability $P(E, \{m_i\})$ is independent of $\{m_i\}$, and equals the probability, $P(E)$, that the J's (with distribution (2)) satisfy

$$\begin{gathered} x_i = \frac{1}{p} \sum_{i_2 < \ldots < i_p} \hat{J}_{i i_2 \ldots i_p} \geqslant 0\,, \quad i = 1, \ldots N\,, \\ E = -\sum_{i_1 < \ldots < i_p} \hat{J}_{i_1 \ldots i_p} = -\sum x_i \end{gathered} \tag{50}$$

The number of solutions will then be given by $2^N P(E)$

We first calculate the probability that the sums in (50) have values $x_1, \ldots, x_N$

$$\begin{aligned} P(x_1 \ldots x_N) = \text{const} \int \prod_{i_1 < \ldots < i_p} \mathrm{d}\hat{J}_{i_1 \ldots i_p} \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left[-\frac{N^{p-1}}{p!} (\hat{J}_{i_1 \ldots i_p})^2\right] \\ \times \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_p \exp\left[i \sum_k \lambda_k \left(x_k - \frac{1}{p} \sum_{i_2 < \ldots < i_p} \hat{J}_{k i_2 \ldots i_p}\right)\right] \end{aligned} \tag{51}$$

The integrations are easily performed with result (for $N \to \infty$)

$$P(x_1 \ldots x_N) = \frac{p^{(N-1)/2}}{\pi^{N/2}} \exp\left[-p\left[\sum_k x_k^2 - \frac{p-1}{Np}\left(\sum_k x_k\right)^2\right]\right] \tag{52}$$

From this distribution it follows that the mean value of x_i is of order $1/\sqrt{p}$ The right-hand side of (46) is of order $\sqrt{p}$, and thus these are solutions of the TAP equations as $p \to \infty$.

First let us calculate the total number of solutions, independent of E This is given by $2^N \int_0^\infty \mathrm{d}x_1 \ldots \mathrm{d}x_N P(x_1 \ldots x_N)$, which, using a gaussian transformation to disentangle the $(\sum_k x_k)^2$ term in (52), can be expressed as

$$\begin{gathered} \mathcal{N} = 2^N \sqrt{N} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} \exp\left[N\left[-\tfrac{1}{2} z^2 + \ln f(z)\right]\right] \\ f(z) = \int_{-z\sqrt{p-1}}^{\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \end{gathered} \tag{53}$$

This can be evaluated by saddle-point methods in the $N \to \infty$ limit We thereby derive that the average number of TAP solutions grows exponentially with N: $\bar{\mathcal{N}} \sim e^{NA}$, where

$$A = \ln 2 - \frac{\mu^2}{2(p-1)} + \ln \int_{-\mu}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2}, \tag{54}$$

and μ is the value of $z\sqrt{p-1}$ at the saddle-point, i.e

$$\mu = \frac{p-1}{\sqrt{2\pi}} \frac{e^{-\mu^2/2}}{\int_{-\infty}^{\mu} dx\, e^{-x^2/2}/2\pi} \tag{55}$$

For $p = 2$, we recover the well-known result that the average number of TAP solutions, at $T = 0$, grows like $e^{0.2N}$ [13, 14]. As p increases so does A and in the $p \to \infty$ limit, it is easily seen that $A \to \ln 2$ Therefore in the $p \to \infty$ SK model the total number of TAP solutions grows like 2^N (more precisely $A \sim 2^N/\sqrt{2 \ln p}$) which means that almost every configuration is a solution of the TAP equations.

What is the physical meaning of these solutions? They, of course, are saddle-points of the TAP free energy However, in our case, one can say more Since the system is frozen, $m_i = \pm 1$, we can interpret a solution as a spin configuration which is a local minimum of the energy This is because the energy can be written as $E = -\sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} m_{i_1} \dots m_{i_p} = -\sum_{i=1}^{N} x_i$, and all the terms x_i are positive If we flip any particular spin, $m_k \to -m_k$, the change in energy is $\delta E = +2x_k > 0$ Now of course not all of these local minima will contribute significantly, in fact we expect that the important configurations will be those with the smallest possible energy [4, 15].

To proceed further we must calculate the average number of solutions, $N(E)$, of a given energy $E = -\sum x_i \equiv -N\varepsilon$ Using the same techniques, this can be cast into the form

$$\overline{\mathcal{N}(E)} = \exp\left[N(\ln 2 + (p-1)\varepsilon^2)\right] \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi\sqrt{2}} \times \exp\left[N\left[-\tfrac{1}{2}\lambda^2 - i\lambda\varepsilon\sqrt{2p} + \ln \int_{-i\lambda}^{\infty} Dx\right]\right] \tag{56}$$

Once again the λ integral can be expanded about the saddle-point $\lambda_s = -i\mu$ as $N \to \infty$;

$$\overline{\mathcal{N}(E)} \sim e^{NA(E)},$$
$$A(E) = (p-1)\varepsilon^2 + \ln 2 + \tfrac{1}{2}\mu^2 - \varepsilon\mu\sqrt{2p} + \ln \int_{-\infty}^{\mu} Dx, \tag{57}$$

where μ is determined by the saddle-point equation

$$\varepsilon\sqrt{2p} - \mu = \frac{1}{\sqrt{2\pi}} \frac{e^{-\mu^2/2}}{\int_{-\infty}^{\mu} Dx} \tag{58}$$

For $p=2$, we recover the results of [13, 14] As $p \to \infty$, we find

$$A(E)=\begin{cases} \ln 2-\varepsilon^2, & \varepsilon \geqslant 0 \\ 0, & \varepsilon<0 \end{cases} \tag{59}$$

This density of TAP solutions, $\mathcal{N}(E) \sim \mathrm{e}^{N[\ln 2-\varepsilon^2]}$, is exactly equal (up to terms which are not exponentially large in N and which do vanish as $p \to \infty$) to the density of energy levels in the random energy model. This is not surprising since it equals the product of the total number of configurations, 2^N, times the probability that a configuration has energy $-N\varepsilon$, $\mathrm{e}^{-N\varepsilon^2}$, times the probability that a configuration is a solution of the TAP equations, $\mathrm{e}^{0 \cdot N}/\sqrt{2 \ln p}$.

To calculate the free energy one would naively sum over all solutions with a Boltzmann-like weight:

$$Z \sim \sum_{\mathrm{sol}} \mathrm{e}^{-E/T}=\int_{E_{\min}} \mathrm{d}E\, \bar{\mathcal{N}}(E)\, \mathrm{e}^{-E/T}, \tag{60}$$

where $E_{\min}$ is the minimum value of the energy for which the number of solutions is $\geqslant 1$ In general (for say $p=2$) this procedure is incorrect [13, 14] It is equivalent to an annealed average over TAP solutions which does not necessarily agree with the correct quenched average of $\ln Z$ One is then led again to the replica method, now applied to TAP solutions [4, 13, 14]. In our case, these complications are unnecessary, since, as we shall see, the different solutions are totally uncorrelated.

We shall calculate the probability $P(E, E', q)$, that two different configurations, $\{m_i\}$ and $\{m_i'\}$, be solutions of the TAP equations with energies E and E' and mutual overlap q Using the same strategy as above, we gauge-transform and introduce variables

$$x_i=\frac{1}{p} \sum_{i_2<\dots<i_p} J_{i i_2 \dots i_p},$$

$$y_i=\frac{1}{p} \sum_{i_2<\dots<i_p} J_{i i_2 \dots i_p} q_{i_2} \cdots q_{i_p}, \tag{61}$$

where $q_i=m_i m_i'$, $q=(1/N) \sum_i q_i$. The TAP equations are equivalent (for $p=\infty$ and taking m_i, m_i' to be ± 1) to

$$\forall i=1, \dots, N, \quad x_i>0, \quad y_i>0 \tag{62}$$

Thus we are led to compute the probability that x_i and y_i, defined above, take specific values, averaging over the J's This yields

$$\begin{aligned} P(x_1 \dots x_N, y_1 \dots y_N, q_i) = {} & \mathrm{const} \times \int \prod_k \mathrm{d}\lambda_k\, \mathrm{d}\mu_k \exp\left[i \sum_k (\lambda_k x_k+\mu_k y_k)\right] \\ & \times \exp\left[-\frac{1}{4p}\left\{\sum_k (\lambda_k^2+\mu_k^2+2\lambda_k \mu_k q^{p-1})\right.\right. \\ & \left.\left. +\frac{p-1}{N} \sum_{k \neq l} (\lambda_k \lambda_l+\mu_k \mu_l q_k q_l+\lambda_k q_k \mu_l q^{p-1})\right\}\right]. \end{aligned} \tag{63}$$

This expression is rather unwieldy, however as $p\to\infty$, it drastically simplifies If $|q|<1$ the q^{p-1} terms vanish, and the distribution factorizes.

$$P(x_1\ldots x_N, y_1\ldots y_N, q)\sim P(x_1\ldots x_N)P(y_1\ldots y_N)\quad(|q|<1) \tag{64}$$

If $q=1$, then all but a vanishing fraction of the q_i's are equal and

$$P(x_1\ldots x_N, y_1\ldots y_N, q=1)\sim\prod_k \delta(x_k-y_k)P(x_1\ldots x_N)\,, \tag{65}$$

which simply means that we are considering the same configuration $\{m_i\}=\{m_i'\}$

Therefore if $\{m_i\}$ and $\{m_i'\}$ are macroscopically different solutions, they are totally uncorrelated We can then argue, as in sect 2, that the fluctuation of $\mathcal{N}(E)$ about its mean $\bar{\mathcal{N}}(E)$ is negligible (of order $1/\sqrt{\mathcal{N}(E)}$) as long as $-\sqrt{\ln 2}<E/N<0$ where $\bar{\mathcal{N}}(E)$ is exponentially large, and one can take $\mathcal{N}(E)$ to equal $\bar{\mathcal{N}}(E)$ Thus we apply (60):

$$e^{-\beta F}\sim\int_{-\sqrt{\ln 2}}^{0}d\varepsilon\,\exp\left[N(\ln 2-\varepsilon^2-\beta\varepsilon)\right] \tag{66}$$

Since $\beta>2\sqrt{\ln 2}$ (recall that we are in the low-temperature phase since we have assumed the system to be frozen with $m_i=\pm1$), the integral is dominated by the minimal value of ε, yielding $F/N=-\sqrt{\ln 2}$, which is the correct result.

The calculation can easily be generalized for nonvanishing magnetic field h The distribution $P(x_1,\ldots,x_N)$ is independent of h. However we must now calculate the total number of solutions of a given magnetization m, and energy $E=-\sum_i x_i-hm$ This equals:

$$N(E,m)=\binom{N}{\frac{1}{2}(N+m)}\times P(E+hm).$$

By arguments identical to those presented above, one can then compute the free energy in the presence of a field, which will coincide with (37)

We can proceed further and ask which of the many TAP solutions actually contribute to the canonical average (66) These must have a free energy

$$F=-N\sqrt{\ln 2}+\hat{F}, \tag{67}$$

where $\hat{F}$ is finite (relative to N) The number of solutions having this free energy F behaves as

$$\mathcal{N}(\hat{F})\sim e^{2\sqrt{\ln 2}\hat{F}}=e^{\hat{F}/T_c} \tag{68}$$

Of course this formula is only valid in the region where $\hat{F}\gg1$, so that the number of solutions between $\hat{F}$ and $\hat{F}+\delta\hat{F}$ be large, and that the fluctuations of $\mathcal{N}$ be negligible These solutions can be identified as pure states of weights

$$P_s=\frac{e^{-\beta F_s}}{\sum_{s'}e^{-\beta F_{s'}}}=C\,e^{-\beta F_s}\,, \tag{69}$$

where C is a temperature-dependent constant From (68) we can obtain the number of states with a given $\hat{F} \gg 1$, which corresponds to a given $P \ll 1$ This equals

$$f(P) \sim \sum_s \delta(P_s - P)$$
$$\sim \text{const} \times \sum_s e^{F_s/T} \delta\left(F_s + T \ln \frac{P}{C}\right)$$
$$\sim \text{const} \times P^{-1-T/T_c} \tag{70}$$

This confirms the asymptotic behaviour for small P of the number of pure states with a given weight P given in (39), and thus the temperature dependence of the (infinite) total number of pure states (40), which is dominated by the states with vanishing weights

We therefore have a clear picture of the ($p = \infty$) spin glass phase, in which the pure states, free energy valleys, can be identified with the minimal energy solutions of the TAP equations We can even say something about the overlaps between these states Since the probability distribution for 2 different solutions factorizes and is independent of the overlap q of the solutions (if $q < 1$), it follows that the mean number of pairs of solutions with overlap q is $\mathcal{N}(q) = 2^N 2^N P(q)$, where $P(q)$ is the probability that, chosen at random, two configurations have overlap q $P(q)$ clearly has a binomial distribution,

$$P(q) = 2^{-N} \binom{N}{\frac{1}{2}N(1+q)},$$

and therefore

$$\mathcal{N}(q) \sim 2^N \, e^{-Nq^2} \sqrt{\frac{N}{2\pi}}, \tag{71}$$

which is highly peaked about $q = 0$ Therefore the overlap between any two solutions can only be· 1 if the solutions are macroscopically indistinguishable, $O(1/\sqrt{N})$ if they are distinguishable In that case the distribution function $P(q)$ will have just two δ-function peaks at $q = 1$ and at $q = 0$. A more careful calculation could probably yield the (temperature-dependent) weights of these δ-functions

5. Conclusions

The infinite range Ising model with p-spin quenched random interactions is a natural generalization of the SK model, which exhibits, at low temperature, spin glass behaviour For $p \to \infty$, it can be solved exactly because of its equivalence to the random energy model We have shown that the exact solution can also be obtained using the techniques that have been applied to the $p = 2$ SK model (and which have produced much of our theoretical understanding of the spin glass phase)

Using the replica method, we have seen that Parisi's ansatz for replica symmetry breaking is valid in the $p \to \infty$ model The average order parameter function $q(x)$

that we obtain is simply a step function, which means that the first breaking of replica symmetry is exact in this case. The physical interpretation of replica symmetry breaking as a description of the breakdown of ergodicity, and the existence of an infinite number of pure states α with ultrametric topology is confirmed Indeed, starting from this interpretation, we have computed the mutual entropy of the pure states $-\sum_\alpha P_\alpha \ln P_\alpha$, which gives exactly the leading finite-N corrections to the entropy in the low-temperature phase, in accordance with the direct computation within the framework of the random energy model

The TAP equations are particularly simple in this model since macroscopically distinct TAP solutions are uncorrelated Hence they can be solved directly without introducing replicas The number of solutions of the TAP equations with free energy f is zero for $f<f_{\min}=-\sqrt{\ln 2}$, and exponentially large for $f>f_{\min}$ We find also confirmation, in this case, of the fact that the canonical average over solutions of TAP equations is dominated by the ones which have free energy $f=f_{\min}$, which can then be identified with the pure states of the system [4]

Since the $p\to\infty$ SK model is so much simpler than the $p=2$ model while it still retains most of the basic properties of a spin glass (especially the existence of an infinite number of pure states unrelated by a symmetry), we think that it constitutes a good starting point for further investigation

Our first suggestion would be to study the large-p expansion of the infinite range model One could calculate the free energy and the function $q(x)$ in an expansion about $p=\infty$ It appears that the corrections are of order e^{-p}, and thus the expansion might be rapidly convergent This might provide analytic insight into the precise structure of the order parameter $q(x)$ for finite p

Another interesting problem for which the $p\to\infty$ model could provide a simple starting point is that of fluctuations about mean field theory, namely the treatment of the model in finite dimensions In this regard we would like to point out that the masses that will appear in the propagators of the perturbation theory about mean field have significant p dependence Since the energy is proportional to Q_{ab}^p, the second derivative with respect to Q is either of order p^2 (if $Q=1$) or zero (if $Q<1$) Therefore it is probably possible to take into account consistently the fluctuations (which will be of order $1/p^2$), while neglecting the finite-p corrections to the mean field theory (which are of order e^{-p}) This might yield an extremely simple perturbation theory, which could be analysed to determine the critical behaviour of the theory

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Appendix A

Here we show that the general kth-order replica symmetry breaking produces the solution derived above At kth-order the matrices λ_{ab} and Q_{ab} are given in terms of the parameters $\lambda_0, \lambda_1, \quad \lambda_k$; $q_0, q_1 \quad q_k$ and $m_0 = n, m_1, \ . \ , m_k, m_{k+1} = 1$, which determine the value of λ_{ab} and Q_{ab} at each level of the tree (fig. 1) and the number of its branches at each generation

The saddle-point equations will always yield

$$\lambda_i = \tfrac{1}{2}\beta^2 p q_i^{p-1} , \tag{A.1}$$

which implies that $\lambda_i \to 0$ (if $q_i < 1$) or $\lambda_i \to \infty$ (if $q_i = 1$) If all the q's are <1 then all the λ's vanish and we will recover the symmetric, high-temperature solution Assume therefore that $q_0 \leqslant q_1 \leqslant \ \cdots \leqslant q_{k-1} < q_k = 1$ We then search for the maximum of $G(q_i, \lambda_i, m_i)$, defined by (23), which in this case equals

$$\frac{1}{n} G(q_i, \lambda_i, m_i) = \sum_{l=0}^{k} (m_l - m_{l+1})[\tfrac{1}{4}\beta^2 q_l^p - \tfrac{1}{2} q_l \lambda_l] + \frac{1}{n} S(\lambda_i, m_i) , \tag{A 2}$$

where $S(\lambda_i, m_i)$ is given by

$$\exp S(\lambda_i, m_i) = \mathrm{Tr}_{(\sigma^a)} \exp \left[\frac{1}{2} \sum_{a \neq b} \lambda_{ab} \sigma_a \sigma_b + \beta h \sum_a \sigma_a \right] \tag{A 3}$$

The evaluation of S is a classic exercise within Parisi's replica symmetry breaking scheme. Using the methods of [6, 16] we find

$$\exp S(\lambda_i, m_i) = \mathrm{e}^{-n\lambda_k/2} \int \mathrm{D}z_0 \left\{ \int \mathrm{D}z_1 \cdots \left[\int \mathrm{D}z_{k-1} I_k^{m_{k-1}/m_k} \right]^{m_{k-2}/m_{k-1}} \cdots \right\}^{m_0/m_1} ,$$
$$I_k \equiv \int \mathrm{D}z_k \, \mathrm{ch}^{m_k} \left(\sum_{l=0}^{k} z_l \sqrt{\tilde{\lambda}_l} + \beta h \right) , \tag{A 4}$$

where $\tilde{\lambda}_0 \equiv \lambda_0$, $\tilde{\lambda}_l \equiv \lambda_l - \lambda_{l-1} (l \geqslant 1)$ and $\mathrm{D}z \equiv \mathrm{d}z \, \mathrm{e}^{-z^2/2}/\sqrt{2\pi}$

Although S is rather complicated it simplifies considerably in our case where $\lambda_k \to \infty$ and $\lambda_{i<k} \to 0$ We need only expand S to first order in $\lambda_{i<k}$. We first consider the innermost integral in (A 3)·

$$\begin{aligned} I_k = & \int \mathrm{D}z_k \, \mathrm{ch}^{m_k}(z_k \sqrt{\tilde{\lambda}_k} + \beta h) \\ & + m_k \left(\sum_{l=0}^{k-1} z_l \sqrt{\tilde{\lambda}_l} \right) \int \mathrm{D}z_k \, \mathrm{ch}^{m_k}(z_k \sqrt{\tilde{\lambda}_k} + \beta h) \, \mathrm{th}\,(z_k \sqrt{\tilde{\lambda}_k} + \beta h) \\ & + \tfrac{1}{2} m_k \left(\sum_{l=0}^{k-1} z_l \sqrt{\tilde{\lambda}_l} \right)^2 \int \mathrm{D}z_k \, \mathrm{ch}^{m_k}(z_k \sqrt{\tilde{\lambda}_k} + \beta h) \\ & \times [1 + (m_k - 1) \, \mathrm{th}^2\,(z_k \sqrt{\tilde{\lambda}_k} + \beta h)] + \mathrm{O}(\tilde{\lambda}_l^{3/2}) \end{aligned} \tag{A 5}$$

For large λ_k this yields

$$I_k = C \left[1 + m_k \sum_{l=0}^{k-1} z_l \sqrt{\tilde{\lambda}_l} \, \mathrm{th}\,(m_k \beta h) + \tfrac{1}{2} m_k^2 \left(\sum_{l=0}^{k-1} z_l \sqrt{\tilde{\lambda}_l} \right)^2 \right] ,$$
$$C = 2^{1-m_k} \, \mathrm{e}^{m_k^2 \lambda_k/2} \, \mathrm{ch}\,(m_k \beta h) \tag{A 6}$$

The next step consists of raising I_k to the power m_{k-1}/m_k and integrating over z_{k-1} This yields

$$I_{k-1}=\int Dz_{k-1}(I_k)^{m_{k-1}/m_k}=C^{m_{k-1}/m_k}\left[1+m_{k-1}\left(\sum_{l=0}^{k-2} z_l\sqrt{\tilde{\lambda}_l}\right)\mathrm{th}\,(m_k\beta h)\right.$$

$$\left.+\tfrac{1}{2}m_{k-1}\left\{\left(\sum_{l=0}^{k-2} z_l\sqrt{\tilde{\lambda}_l}\right)^2+\tilde{\lambda}_{k-1}\right\}\left\{m_k+(m_{k-1}-m_k)\,\mathrm{th}^2\,(m_k\beta h\right\}\right] \quad (A.7)$$

This can then be iterated, finally yielding

$$e^{n\lambda_k/2+S(\lambda_l,m_l)}=I_0=C^{m_0/m_k}\left[1+\tfrac{1}{2}m_0\sum_{l=0}^{k-1}\tilde{\lambda}_l\{m_k+(m_0-m_k)\,\mathrm{th}^2\,(m_k\beta h)\}\right]. \quad (A\,8)$$

This enables us to determine $G(q_l, \lambda_l, m_l)$ to the desired order

$$\frac{1}{n}G(q_l,\tilde{\lambda}_l,m_l)\approx\sum_{l=0}^{k}(m_l-m_{l+1})[\tfrac{1}{4}\beta^2q_l^p-\tfrac{1}{2}q_l\sum_{r=0}^{l}\tilde{\lambda}_r]$$

$$-\frac{1}{2}\sum_{l=0}^{k}\tilde{\lambda}_l+\frac{1}{2}\sum_{l=0}^{k-1}\tilde{\lambda}_l[m_k+(m_l-m_k)\,\mathrm{th}^2\,(m_k\beta h)]$$

$$+\frac{1}{m_k}[(1-m_k)\ln 2+\tfrac{1}{2}m_k^2\tilde{\lambda}_k+\ln\mathrm{ch}\,(m_k\beta h)] \quad (A\,9)$$

We can now examine the saddle-point equations. The variation with respect to $\tilde{\lambda}_l$ yield

$$q_k=1 \quad (l=k)\,,$$

$$1+\sum_{r=l}^{k}q_r(m_r-m_{r+1})=m_k+(m_l-m_k)\,\mathrm{th}^2\,(m_k\beta h) \quad (l<k) \quad (A\,10)$$

If we solve this equation successively for $l=k-1$, $l=k-2$, , $l=1$, we find

$$q_{k-1}=q_{k-2}=\cdots=q_0=\mathrm{th}^2\,(m_k\beta h)$$

Thus we recover the previous saddle-point associated with the first stage of replica symmetry breaking. Hence there are no new saddle-points appearing at the higher stages of replica symmetry breaking

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