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LARGE-N REDUCTION IN SPIN SYSTEMS AND GRIFFITHS SINGULARITIES

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We apply the quenched Eguchi-Kawai reduction procedure to N-component spin models. We first recover the equivalence of the $O(N)$ symmetric Heisenberg model with the spherical model at large N, and we extend it to the case where a quenched random external field is present. When the random field has a gaussian distribution, we show that Griffiths singularities disappear as $N \to \infty$.

1. Introduction

Field theories with symmetry group $O(N)$ (or $U(N),\ldots$) become quite simple when N becomes large. This large-N limit (and $1/N$ expansion) has been extensively studied in spin models and in matrix models. In the first case, where the fields belong to an N-dimensional representation of the group, one can generally solve the model for $N \to \infty$. As for the second case, where the fields are in an N^2-dimensional representation, there are also many simplifications when N gets large, although an exact solution is still generally out of reach. Last year, an important step was taken in this direction: starting from the original idea of Eguchi and Kawai [1], it has been shown that, in the large-N limit, gauge theories can be reduced to one space-time point, provided one uses quenched momentum variables [2–4].

This reduction can also be applied to spin systems. It has been recently used in this case by Goldschmidt [5] who introduces replicas to perform the quenched average over the momenta in the reduced model. We shall see that this integration over momenta is not needed, so that one can avoid the introduction of replicas and the usual intricacies associated with them.

Furthermore, we can make a special choice of quenched momenta such that the reduced theory (which is a model involving only one N-component spin) be equivalent to a theory of N spins having, each, only one component at each point of

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a d-dimensional lattice. We then have to study the usual thermodynamic limit of this last model.

An interesting problem to which this reduction technique can be applied is the problem of Heisenberg spins in an external random field. This is one of the simplest statistical mechanics system with randomness, and it exhibits striking effects: it is generally believed that the critical behavior of a spin system in a random field in d dimensions is the same as that of the pure system in $(d-2)$ dimensions [6–9]. However, as has been emphasized by Parisi [10], the general validity of this result is somewhat obscured by the existence of Griffiths singularities. From the work of Griffiths on the randomly diluted Ising system [11], one expects that there will appear, in a random system in d dimensions, essential singularities in physical quantities at and below the critical temperature of the pure d-dimensional system. The intuitive explanation of these singularities in a diluted system is the following: given a region of space as large as one wants, there exist configurations (which appear with very small probabilities) such that this region is impurity free, and develops a spontaneous magnetization. These very large impurity free clusters give rise to singularities in the free energy.

The large-N limit is a simple non-perturbative approximation scheme where one can try to understand these singularities. Using the quenched reduction technique, one finds that the large-N limit of the classical Heisenberg model in a random field is equivalent to the spherical model in a random field. In this spherical model, we show that, when the probability distribution of the random magnetic field is gaussian, Griffiths singularities do not appear. This result is confirmed by mean-field theory arguments, which indicate that these singularities vanish exponentially when $N \to \infty$ for N component spin systems in a gaussian random field.

In sect. 2, we explain the quenched reduction in spin systems, with special emphasis on a special choice of momenta which allows us to recover the known results. This procedure is then applied to spin systems in a random field.

Sect. 3 is devoted to an analysis of Griffiths singularities: it is first shown that they are absent in the spherical model in a random field; we then study the mean-field theory arguments of Parisi and extend them to N-component spins.

These results are summarized in sect. 4.

2. Large-N reduction for spin models

The aim of this section is to show that the large-N limit of a d-dimensional spin system can be most easily analyzed by means of the following two-steps procedure: reduction to one space-time point, followed by a mapping of this reduced model (one N-component spin) onto a model of one-component spins on a d-dimensional lattice. The first step is not very surprising since it is well-known that for large N, the functional integral is generally dominated by the contribution of a uniform saddle-point. As for the second step, its meaning can be somehow clarified by quoting the result in a simple case: when applied to the classical Heisenberg model, it leads to a system of N one-component spins S_i at each site of a d-dimensional lattice with the global constraint $\sum_i S_i^2 = N$. So we recover the fact that the large-N limit of the Heisenberg model is the spherical model [12,13]. What we shall see in the following is that this kind of equivalence exists in more general models, and that it remains valid in the presence of a random magnetic field. The proof we shall give here relies on the identification of the dominant diagrams in the perturbative expansions of the original model and of the reduced one. It can be carried out both in the high-temperature phase and in the low-temperature phase, showing that both models are truly equivalent. Let us also mention that another derivation can be obtained from the stochastic quantization method [17], which allows to incorporate the random field in a simple way.

In order to keep the notation simple, we shall consider the case of the $(|\phi|^2)^2$ theory. $\phi(x_i)$ is an N component complex vector field* sitting at the nodes of a hypercubic d-dimensional lattice $\mathcal{L}$ of size $\mathcal{N} = L^d$. (We choose the units such that the lattice spacing be equal to one.) The partition function is

$$Z_{\mathcal{N}}(\beta, N) = \int \mathcal{D}\phi \exp\Bigg\{ \tfrac{1}{2}\beta \sum_{(i,j)} \sum_{a=1}^{N} \left(\phi_a^*(x_i)\phi_a(x_j) + \phi_a^*(x_j)\phi_a(x_i) \right) - m^2 \sum_{i,a} |\phi_a(x_i)|^2 - \frac{g}{N} \sum_i \Big(\sum_a |\phi_a(x_i)|^2 \Big)^2 \Bigg\}, \tag{1}$$

where the interaction $\sum_{(i,j)}$ is between nearest neighbours.

The corresponding reduced model is given by [3–5]

$$Z_{\mathrm{R}}(\beta, N, P_a) = \int \mathrm{d}\phi \exp\Bigg\{ \tfrac{1}{2}\beta \sum_{a=1}^{N} \sum_{\mu=1}^{d} (2\cos P_a^{\mu}) |\phi_a|^2 - m^2 \sum_a |\phi_a|^2 - \frac{g}{N} \Big(\sum_a |\phi_a|^2 \Big)^2 \Bigg\}, \tag{2}$$

where we have introduced N momenta $P_a = (P_a^1, \ldots, P_a^d)$ which must be integrated over the Brillouin zone $[0, 2\pi]^d$. The result is then the following: the free energy per site of the original problem (1) is equal, in the large-N limit, to the quenched free energy of the model reduced to one site (2), that is:

$$\lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \log Z_{\mathcal{N}}(\beta, N) \overset{N\to\infty}{\sim} \int \prod_a \frac{\mathrm{d}^d P_a}{(2\pi)^d} \log Z_{\mathrm{R}}(\beta, N, P_a). \tag{3}$$

* We use here a complex field for notational convenience. It is easily seen that the argument can be applied for real fields as well. We give the general result in (9)–(11).

A simple way to prove this equivalence (3) is to notice that the diagrammatic expansion of the quenched reduced model gives back, exactly, the bubble diagrams that dominate in the diagrammatic expansion of the original model, with the same weight and the same momentum integrations. But this equivalence goes beyond perturbation theory since both theories verify the same whole set of Schwinger-Dyson equations for large $N^\star$ (This is easily seen by using the well-known factorization properties of correlation functions at large N). All these results are analogous to what is found in gauge theories where the reduced system reproduces the planar diagrams [3–5].

However, it is not necessary to integrate over the quenched momenta in (3) [4]. In fact, one can choose, instead, specific values of the P_a which densely and uniformly cover the Brillouin zone in the large-N limit. The reason for this is that the function $\log Z_R(\beta, N, P_a)$ is symmetric under any permutation of the P_a. Therefore, it is equal to the average over permutations $\langle \log Z_R(\beta, N, P_a)\rangle_{\mathrm{perm}}$. But for large N this average tends to the statistical average of P_a uniformly distributed in the Brillouin zone, precisely used in (3). Hence one simply gets:

$$\lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \log Z_{\mathcal{N}}(\beta, N) \overset{N\to\infty}{\sim} \log Z_R(\beta, N, P_a), \tag{4}$$

provided the P_a are chosen so as to cover, densely and uniformly, the Brillouin zone as $N \to \infty$. In fact, in this simple case of spin systems, this prescription (4) is even better than the original one (3) since it remains valid for diagrams with more than N loops.

We shall now make use of this possibility of choosing the momenta, to obtain a mapping of the internal degrees of freedom onto points in momentum space: we shall map the N-component model reduced to one point, as given in (2), onto a one-component model on a d-dimensional lattice with N sites. Then the large-N limit will just be the thermodynamic limit for this last model. We may suppose that N is of the form: $N = (L')^d$. Then we can choose to characterize each value of the vector indices "a" by a point on a hypercubic lattice $\mathcal{L}'$ of size $(L')^d$ and a lattice spacing equal to one, namely:

$$a = (a_1, a_2, \dots, a_d), \tag{5}$$

the a_i being integer numbers such that $0 \leqslant a_i \leqslant L' - 1$. Our choice of P_a is the following:

$$P_a^\mu = P^\mu_{(a_1, a_2, \dots, a_d)} = 2\pi \frac{a_\mu}{L' - 1}. \tag{6}$$

This choice is obviously of the form required for the validity of (4), since we have

* Of course, we keep to Green functions which are invariant under the symmetry group (here $O(2N)$).

just divided the Brillouin zone into $(L')^d$ identical cubes, each one containing one of the P_a. We can now consider the P_a as wave vectors associated with the lattice $\mathcal{L}'$, hence the indices a label these wave vectors. Let us identify ϕ_a as Fourier transforms of a new field $\tilde{\phi}$ sitting at the nodes r_i of the lattice $\mathcal{L}'$:

$$\tilde{\phi}(r_i) = \frac{1}{\sqrt{N}} \sum_a \mathrm{e}^{-iP_a \cdot r_i} \phi_a . \tag{7}$$

In the expression (2) of the reduced partition function Z_{R}, we perform a change of variables from ϕ_a to $\tilde{\phi}$; the result is:

$$Z_{\mathrm{R}}(\beta, N) = \int \prod_1^{L'^d} \mathrm{d}\tilde{\phi}(r_i) \exp\left\{ \tfrac{1}{2}\beta \sum_{(i,j)} \left(\tilde{\phi}^*(r_i)\tilde{\phi}(r_j) + \tilde{\phi}^*(r_j)\tilde{\phi}(r_i) \right) \right.$$

$$\left. - m^2 \sum_i \left|\tilde{\phi}(r_i)\right|^2 - \frac{g}{N}\left(\sum_i \left|\tilde{\phi}(r_i)\right|^2 \right)^2 \right\}, \tag{8}$$

and the summation $\sum_{(i,j)}$ is the sum over nearest neighbours. We have obtained the following result: the free energy per degree of freedom of the initial $(g/N)(|\phi|^2)^2$ model is equal, in the large-N limit, to the free energy per site of a system of one-component field $\tilde{\phi}$, with a delocalized potential $(g/N)(\sum_i |\tilde{\phi}(r_i)|^2)$.

As this derivation relies only on the quenched Eguchi-Kawai reduction with the special choice of momenta (6), it can be generalized to other models. For instance let us consider the general class of models that were studied by Halpern [14] with a saddle-point method:

$$Z_{\mathcal{N}}(\beta) = \int \prod_{a=1}^{N} \mathcal{D}\phi_a \exp\left\{ \tfrac{1}{2}\beta \sum_{i,j=1}^{\mathcal{N}} v_{ij} \left(\sum_{a=1}^{N} \phi_i^a \phi_j^a \right) - N \sum_i V\left(\frac{g}{N} \sum_a \phi_i^a \phi_i^a \right) \right\}, \tag{9}$$

where v_{ij} is an arbitrary (translation invariant) interaction, and V an arbitrary potential. The corresponding reduced partition function is:

$$Z_{\mathrm{R}}(\beta, P_a) = \int \prod_{a=1}^{N} \mathrm{d}\phi_a \exp\left\{ \tfrac{1}{2}\beta \sum_a v(P^a) \phi^a \phi^a - NV\left(\frac{g}{N} \sum_a \phi^a \phi^a \right) \right\}, \tag{10}$$

and with the choice (6) for the momenta, it can be mapped onto a model on a $N = (L')^d$ lattice:

$$Z_{\mathrm{R}}(\beta) = \int \mathcal{D}\tilde{\phi} \exp\left\{ \tfrac{1}{2}\beta \sum_{i,j=1}^{N} v_{ij} \tilde{\phi}_i \tilde{\phi}_j - NV\left(\frac{g}{N} \sum_i \tilde{\phi}_i^2 \right) \right\}. \tag{11}$$

In the large-N limit one finds:

$$\lim_{\mathfrak{N}\to\infty} \frac{1}{\mathfrak{N}N} \log Z_{\mathfrak{N}}(\beta, N) \overset{N\to\infty}{\sim} \frac{1}{N} \log Z_{\mathrm{R}}(\beta)\,. \tag{12}$$

This shows the equivalence of the three problems:

the N ($\to\infty$)-component model with a local, $O(N)$ invariant potential $\Sigma_i V([g/N]\Sigma_i \phi_i^2)$ in the thermodynamic limit (9);

the N ($\to\infty$)-component model, with potential V, reduced on one lattice site, with quenched momenta (10);

the one-component model with delocalized potential $V([g/N]\Sigma_i \tilde{\phi}_i^2)$, in the thermodynamic limit (11).

Finally, let us notice that it is very easy to solve these models starting from the one component form (11), by defining an auxiliary field $\chi = \Sigma_i \tilde{\phi}_i^2$. After a trivial calculation one obtains the usual gap equations of this model (see [14, 5]). It must however be emphasized that, although this method allows easy computation of the leading behavior for $N\to\infty$, it does not give, as it stands, the $1/N$ corrections.

To end this section, we point out that the quenched reduction can also be applied to a spin system in a random field. This will imply that the large-N limit of the Heisenberg model in a random, spherically symmetric, field, is nothing but the spherical model in a random field. For definiteness, we shall stay in the framework of the $(|\phi|^2)^2$ theory*. Let us suppose that this system is placed in a magnetic field $\boldsymbol{h}(x_i)$ which is a quenched random field variable with a probability distribution $\mathrm{d}\mu\,[\boldsymbol{h}]$. The relevant quantity is then the average of the free energy over the field configurations:

$$\bar{F} = \lim_{\mathfrak{N}\to\infty} \frac{1}{\mathfrak{N}} \int \mathrm{d}\mu[\boldsymbol{h}] \log Z_{\mathfrak{N}}[\boldsymbol{h}]\,. \tag{13}$$

We shall only study the simple problem where the magnetic field has a gaussian distribution:

$$\mathrm{d}\mu[\boldsymbol{h}] = \prod_i \left[\mathrm{d}^{(N)}\boldsymbol{h}_i (\pi w)^{-N/2} \exp\left(-\frac{1}{w} \sum_i |\boldsymbol{h}_i|^2 \right) \right]. \tag{14}$$

In this case one can derive a diagrammatic expansion for $\bar{F}$: the diagrams for $\bar{F}$ are the usual connected vacuum graphs of the $(|\phi|^2)^2$ theory, with "w insertions" on certain propagators: one has two types of propagators:

normal propagator:

$$= \frac{1}{\beta v(p)} \delta_{ab},$$

w insertion: 

$$= \frac{1}{\beta v(p)} w \frac{1}{\beta v(p)} \delta_{ab},$$

* From now on, the field is considered as real.

and the only constraint on the position of the insertion is that, if one cuts the diagram along the crosses, one still has a connected diagram. Since the insertions are diagonal in internal space, the dominant diagrams in the large-N limit are still bubble chains, with a certain number of w insertions subject to the above-stated rule.

The same diagrammatic expansion, with the same rule for w insertions, is obtained from the following reduced model:

$$Z_{\mathrm{R}}(\boldsymbol{h}, P_a) = \int \mathrm{d}\boldsymbol{\phi} \exp\left[\tfrac{1}{2}\beta \sum_{a=1}^{N} \sum_{\mu=1}^{d} (2\cos P_a^{\mu})(\phi_a)^2 \right.$$
$$\left. - m^2 \sum_a (\phi_a)^2 - \frac{g}{N}\left(\sum_a (\phi_a)^2\right)^2 + \left(\sum_a h_a \phi_a\right)\right], \qquad (15)$$

where h_a is just the N-component field on one site, with the gaussian distribution:

$$\mathrm{d}\mu_{\mathrm{R}}[h] = \mathrm{d}^{(N)}\boldsymbol{h}\,(\pi w)^{-N/2} \exp(-\boldsymbol{h}^2/w) \qquad (16)$$

and the average over $\boldsymbol{h}$ is quenched, which means that the reduced mean free energy is

$$\bar{F}_{\mathrm{R}} = \int \mathrm{d}\mu_{\mathrm{R}}[h] \int \prod_a \frac{\mathrm{d}^d P_a}{(2\pi)^d} \log Z_{\mathrm{R}}(\boldsymbol{h}, P_a). \qquad (17)$$

The result is exactly analogous to the case where there is no random field. Here also, the result is valid to all orders in the perturbative expansions, both in the high-temperature and in the low-temperature phases:

$$\frac{1}{N}\bar{F}(\beta, N) \overset{N\to\infty}{\sim} \frac{1}{N}\bar{F}_{\mathrm{R}}(\beta, N). \qquad (18)$$

We can then choose the momenta P_a as in (6) to get a one-component model, with a delocalized potential $(\Sigma_i(\tilde{\phi}_i)^2)^2$, in a random field. The generalization to other rotation invariant potentials is straightforward.

3. Spherical model in a random field and Griffiths singularities

As we have seen before, the large-N limit of the classical Heisenberg model in a random field is thermodynamically equivalent to the spherical model in a random field. For this reason, we shall first study this spherical model where we find that Griffiths singularities do not appear. This suggests that these singularities vanish for N large. Then we shall show that our result can be simply understood in the mean-field theory approximation.

The spherical model is a simple model exhibiting a second-order phase transition that can be solved in any dimension [12], and as such has been much studied. In the presence of an external random field, the first evidence of the shift of critical dimensionality $d \to d-2$ was found by Lacour-Gayet and Toulouse [6] in a problem which is formally equivalent to the spherical model: the ideal gas Bose condensation. Recently the random field spherical model has been studied by Pastur [15] and by Hornreich and Schuster [16]. In fact the authors of ref. [16] started from the large-N Heisenberg model, using the replica trick and a saddle-point method. This involves the assumption that the saddle-point is symmetric in replica space. As it is not clear to us whether this method may hide eventual Griffiths singularities, we shall avoid the introduction of replicas.

The partition function of the spherical model in a random field is:

$$Z_s[h,\beta] = \int_{\mathcal{S}} \prod_{q=1}^{N} \mathrm{d}\phi_q \exp\left\{ \tfrac{1}{2}\beta \sum_{q=1}^{N} v(q)\phi_q^2 + \beta \sum_q h_q \phi_q \right\}, \tag{19}$$

where q are momenta on the d-dimension lattice, ϕ_q are the Fourier transforms of field variables and $v(q)$ is the Fourier transform of the interaction potential. The integration volume is restricted to the hypersphere $\mathcal{S}$:

$$\sum_q \phi_q^2 = N. \tag{20}$$

Although our argument can be used directly on (19), it is easier to study rather the "mean spherical model", where the constraint (20) is implemented on average only. Therefore one introduces a chemical potential λ, and the partition function is:

$$Z_{\mathrm{MS}}[h,\lambda,\beta] = \frac{1}{A_N} \int \prod_{q=1}^{N} \mathrm{d}\phi_q \exp\left\{ \tfrac{1}{2}\beta \sum_q v(q)\phi_q^2 + \beta \sum_q h_q \phi_q - \lambda\left(\sum_q \phi_q^2 - N \right) \right\}, \tag{21}$$

where A_N is a normalization constant, and the chemical potential λ is chosen such that:

$$\frac{\partial \log Z_{\mathrm{MS}}[h,\lambda,\beta]}{\partial \lambda} = 0. \tag{22}$$

The relation between the spherical model and the mean spherical model is exactly analogous to that which exists between the canonical and the grand canonical ensembles in statistical mechanics. So one expects that the two models are equivalent in the thermodynamic limit, at least at the points where no phase transition occurs.

The fact that this result holds in the presence of a random field has been shown by Pastur [15]. We also notice that, with formulae (21)–(22), the analogy of our problem with the ideal gas Bose condensation studied in [6] and described here with a grand canonical ensemble, is clear.

Let us now state the main problem: the chemical potential computed from (22) is a function of β and of the field configuration h: $\lambda = \lambda_0[h, \beta]$, and the averaged free energy that one wants to compute is:

$$\bar{F} = \lim_{N\to\infty} \frac{1}{N} \int \mathrm{d}\mu[h] \log Z_{\mathrm{MS}}[h, \lambda_0[h,\beta], \beta] \,. \tag{23}$$

Of course, it is difficult in (23) to keep track of the h dependence of λ_0, since this is given by an implicit equation (22). The authors of refs. [6, 15] therefore introduce a mean chemical potential which verifies (22) only on average. We shall briefly describe a method which allows us to do the true average as defined in eqs. (22), (23).

We start from the partition function Z_{MS} in (21). The integral over the field ϕ_q is unconstrained, so it is just a gaussian integral that is done easily and gives:

$$Z_{\mathrm{MS}} = \frac{\mathrm{e}^{\lambda N}}{A_N} \exp\left[\sum_{q=1}^{N} \left\{ \frac{1}{2} \frac{\beta^2 h_q^2}{2\lambda - \beta v(q)} - \tfrac{1}{2} \log\left(\frac{2\lambda - \beta v(q)}{2\pi} \right) \right\} \right]. \tag{24}$$

Let us notice that, in order for Z_{MS} to be defined, the chemical potential λ must be in the range $\mathrm{Re}\,\lambda > \frac{1}{2}\beta \max v(q)$, which gives, for nearest neighbour interactions: $\mathrm{Re}\,\lambda > \frac{1}{2}\beta v(0) = \beta d$. We define $z = \lambda/\beta$, and rescale the potential by $\frac{1}{2}$ such that $v(q) = \sum_{\mu=1}^{d} \cos q_\mu$. We get:

$$W[h,z] \equiv \log Z_{\mathrm{MS}} = \sum_q \left\{ \beta z - \tfrac{1}{2}\log 2\beta - \tfrac{1}{2}\log(z - v(q)) + \tfrac{1}{4}\beta \frac{h_q^2}{z - v(q)} \right\}, \tag{25}$$

where we have dropped an arbitrary constant. The constraint equation (22) is $\partial W/\partial z = 0$, which defines the chemical potential $z = z_0[h]$. (Hereafter, we shall not write down the dependence on β explicitly.)

In order to compute the free energy in (23), we need $W[h, z_0[h]]$, which we will write as

$$\begin{aligned} W[h, z_0[h]] &= \int_d^\infty \mathrm{d}z\, W[h,z]\, \delta(z - z_0[h]) \\ &= \int_d^\infty \mathrm{d}z\, W[h,z]\, \delta\left(\frac{\partial W}{\partial z} \right) \left| \frac{\partial^2 W}{\partial z^2} \right| \\ &= \int_d^\infty \mathrm{d}z \int \mathrm{d}\alpha\, W[h,z] \left| \frac{\partial^2 W}{\partial z^2} \right| \exp\left(i\alpha \frac{\partial W}{\partial z} \right). \end{aligned} \tag{26}$$

In deriving (26), we have used the fact that the equation $\partial W/\partial z = 0$ has only one solution $z_0[h]$. On the form (26), one can do the average over h: We shall as before keep to fields with a gaussian distribution:

$$\mathrm{d}\mu[h] = \prod_{q=1}^{N} \left[(\pi w)^{-1/2} \exp\{ -h_q^2/w \} \right]. \tag{27}$$

From (26), we obtain the following average free energy:

$$\begin{aligned}
\bar{F} = \lim_{N\to\infty} \frac{1}{N} \int_d^\infty \mathrm{d}z \int \mathrm{d}\alpha \\
&\times \left\{ N \int \frac{\mathrm{d}^d q}{(2\pi)^d} \left(\frac{1}{2[z - v(q)]^2} + \frac{\frac{1}{4}\beta w}{[z - v(q)]^3 \left(1 + \frac{1}{4} w\beta i\alpha / [z - v(q)]^2\right)} \right) \right\} \\
&\times \left\{ N \int \frac{\mathrm{d}^d q'}{(2\pi)^d} \left(\beta z - \tfrac{1}{2} \log[2\beta(z - v(q'))] \right.\right. \\
&\qquad \left.\left. + \frac{\frac{1}{8}\beta w}{[z - v(q')] \left(1 + \frac{1}{4} w\beta i\alpha / [z - v(q')]^2\right)} \right) \right\} \\
&\times \exp\left\{ N \int \frac{\mathrm{d}^d q''}{(2\pi)^d} \left(i\alpha \left[\beta - \frac{1}{2} \frac{1}{z - v(q'')} \right] \right.\right. \\
&\qquad \left.\left. - \tfrac{1}{2} \log\left[1 + \tfrac{1}{4} w\beta i\alpha / [z - v(q'')]^2\right] \right) \right\}.
\end{aligned} \tag{28}$$

This is a two-dimensional integral which can be computed for $N \to \infty$ by a saddle-point method: one must find the saddle-points of the function:

$$J(\alpha, z) = i\alpha \left[\beta - \tfrac{1}{2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{z - v(q)} \right] - \tfrac{1}{2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \log\left[1 + \frac{w\beta i\alpha}{4(z - v(q))^2} \right]. \tag{29}$$

By writing the stationary equations, one finally finds that the saddle-point must have $\alpha_s = 0$, hence z_s must be a solution of:

$$\begin{aligned}
2\beta &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{z_s - v(q)} + \tfrac{1}{4}\beta w \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{[z_s - v(q)]^2}, \\
\alpha_s &= 0.
\end{aligned} \tag{30}$$

This allows us to compute the free energy in (28), and one finds

$$\bar{F} = \beta z_s - \tfrac{1}{2}\int \frac{d^d q}{(2\pi)^d} \log[2\beta(z_s - v(q))] + \tfrac{1}{8}\beta w \int \frac{d^d q}{(2\pi)^d} \frac{1}{z_s - v(q)}, \quad (31)$$

including the fluctuations around the saddle-point. The equations (30) and (31) summarize the results obtained by computing the "true" free energy defined in eqs. (22), (23). However these equations are precisely those that can be obtained by using the prescription of the average chemical potential [6,15]. So we have found that, in this case, the probability distribution of chemical potentials is well-behaved and the approximation using an average chemical potential gives the correct result. In particular there is no appearance of Griffiths singularities in this problem. From (30) and (31) one can obtain all the relevant information on this system. For instance, from (30), one immediately finds that the lower critical dimension is 4, for w sufficiently small. Indeed, the equation for β is

$$2\beta = \frac{I_1(z_s)}{1 - \tfrac{1}{8} w I_2(z_s)}, \quad (32)$$

where

$$I_1(z) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{z - v(q)}, \qquad I_2(z) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{(z - v(q))^2}.$$

For $d \leqslant 4$, $I_2(z)$ is divergent at $z = d$, which means that the saddle-point $z_s(\beta)$ can be found from (32) for all β: there is no phase transition. For $d > 4$, there is a phase transition if and only if w is smaller than $8/I_2(z = d)$.

More detailed properties of this interesting model, including critical exponents and the value of the Edwards-Anderson order parameter, have been studied in the literature [6,16], so we shall not develop these points here.

In the mean-field theory approximation the different points decouple and one is reduced to the study of a zero-dimensional model. For one-component spins Parisi has shown that the Griffiths singularities are present also in this zero-dimensional case [10]. For completeness we shall briefly give his argument and generalize it to N-component spins. The potential is

$$S = \tfrac{1}{2} m^2 \boldsymbol{\phi} \cdot \boldsymbol{\phi} + \frac{g}{4N} (\boldsymbol{\phi} \cdot \boldsymbol{\phi})^2 + \boldsymbol{h} \cdot \boldsymbol{\phi}. \quad (33)$$

The mean-field solution is the solution $\boldsymbol{\phi}_h$ which is the minimum of this potential; it satisfies the equations:

$$\left(m^2 + \frac{g}{N} \boldsymbol{\phi}_h \cdot \boldsymbol{\phi}_h\right) \phi_h^a = -h^a. \quad (34)$$

The point $m^2 = 0$ corresponds to the critical temperature of the pure system. We are interested in the vicinity of this point, that is m^2 negative and small. Let $\boldsymbol{\phi}_h(i)$, $i=1,\ldots,s$ be the extrema of the potential, solutions of (34), ordered with increasing energy. The true average correlation function is obtained from the absolute minimum of the potential:

$$G=\int \mathrm{d}\mu[\boldsymbol{h}]\boldsymbol{\phi}_h(1)\cdot\boldsymbol{\phi}_h(1). \tag{35}$$

In this case the absolute minimum lies in the region where

$$\boldsymbol{\phi}_h\cdot\boldsymbol{\phi}_h+\frac{m^2N}{g}>0, \tag{36}$$

and the average correlation function is given by:

$$G=\int \mathrm{d}\boldsymbol{h}\exp(-\boldsymbol{h}^2/w)\int \mathrm{d}\boldsymbol{\phi}\,\boldsymbol{\phi}^2\theta(\boldsymbol{\phi}^2+m^2N/g)$$
$$\times\prod_a\delta\left[\left(m^2+\frac{g}{N}\boldsymbol{\phi}^2\right)\phi^a+h^a\right]\left\{\det\left[\left(m^2+\frac{g}{N}\boldsymbol{\phi}^2\right)\delta_{ab}+\frac{2g}{N}\phi_a\phi_b\right]\right\}. \tag{37}$$

On the other hand, the average correlation function G^{DR} associated with the dimensional reduction theory is easily obtained from, for instance, ref. [9]: it is given by the same formula as (37), without the θ function. Hence G^{DR} is incorrect in the case where (34) has several solutions since it computes the sum over the extrema of the potential:

$$G^{\mathrm{DR}}=\sum_{i=1}^{s}\boldsymbol{\phi}_h(i)\cdot\boldsymbol{\phi}_h(i)\times\operatorname{sign}\left\{\det\left[\left(m^2+\frac{g}{N}\boldsymbol{\phi}_h^2(i)\right)\delta_{ab}+\frac{2g}{N}\phi_h^a(i)\phi_h^b(i)\right]\right\}. \tag{38}$$

In the case $N=1$, one sees from (37) that G^{DR} misses the singularity at the point $m^2=0$ which is present in G because of the θ function.

Let us now see what happens for large N:

$$G=\int \mathrm{d}\boldsymbol{\phi}\,\boldsymbol{\phi}^2\theta\left(\boldsymbol{\phi}^2+\frac{m^2N}{g}\right)\left(m^2+\frac{g}{N}\boldsymbol{\phi}^2\right)^{N-1}\left(m^2+\frac{3g}{N}\boldsymbol{\phi}^2\right)$$
$$\times\exp\left[-\frac{1}{w}\boldsymbol{\phi}^2\left(m^2+\frac{g}{N}\boldsymbol{\phi}^2\right)^2\right]. \tag{39}$$

We rescale the field by writing $|\boldsymbol{\phi}|=\sqrt{N}\,r$, which gives

$$G=c_N\int_0^\infty \mathrm{d}r\,r^2\theta(m^2+gr^2)(m^2+3gr^2)\exp\left[-\frac{1}{w}r^2(m^2+gr^2)^2\right]$$
$$\times\exp\left\{(N-1)\left[\tfrac{1}{2}\log r^2+\log(m^2+gr^2)-\frac{1}{w}r^2(gr^2+m^2)^2\right]\right\}. \tag{40}$$

For $N \to \infty$, we evaluate this integral by a saddle-point approximation. The result is the following: for m^2 slightly negative, there are two possible saddle-points:

one at $r^2 = \frac{1}{3}(-m^2/g) \equiv r_0^2$;

one which is the solution r_1^2 of the equation $\frac{1}{2}w = r^2(gr^2 + m^2)^2$.

One finds that $r_1^2 > (-m^2/g)$, and for $(-m^2)$ sufficiently small, the dominant saddle-point is r_1*. This means that the dominant contribution to G around the point $m^2 = 0$ can be obtained from (40) by forgetting the θ function: for large N, the average correlation function G does not develop any singularity around the critical point of the pure theory. In fact, the singular term in (40) vanishes exponentially for $N \to \infty$, relatively to the dominant contribution.

One point that must be emphasized is that all these results were obtained for the problem of a quenched random field with gaussian distribution. It might be possible that they do not remain valid for other types of distributions of the random field.

4. Conclusion

The study of the large-N limit in matrix models seemed to be, at least technically, quite different from what happens in spin systems. The quenched reduction procedure provides a kind of unified framework for these two types of models. One important problem which remains unsolved in the reduction of gauge theories is how to incorporate $1/N$ corrections. From this point of view it is interesting to see how the reduction works in spin systems: in that case one knows what these corrections are by computing fluctuations around the saddle-point. Hence it might be a place where one could try to exhibit a modified reduction scheme that would give the $1/N$ terms correctly.

On the other hand, large-N reduction in spin systems turns out to be interesting in itself. Here we have applied it to random-field problems, and this allowed us to show that Griffiths singularities vanish for $N \to \infty$, at least in the case of an external random field which has a gaussian distribution.

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Note added in proof:

The problem of the spherical model in a random field has also been considered in: M. Shwartz, Phys. Lett. 76A (1980) 408.

* This is true in the vicinity of the critical temperature of the pure system, that is for: $(-m^2) < (\frac{27}{8}gw)^{1/3}$. If $(-m^2)$ becomes larger than this critical value, singularities shall develop again.

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