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The Lévy spin glass transition

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Abstract – We determine the phase transition in the Lévy spin glass. A regularized model where the coupling constants smaller than some cutoff ε are neglected can be studied by the cavity method for diluted spin glasses. We show how to handle the $\varepsilon \to 0$ limit and determine the de Almeida-Thouless transition temperature in the presence of an external field. Contrary to previous findings, in zero external field we do not find any stable replica-symmetric spin glass phase: the spin glass phase is always a replica-symmetry–broken phase.

Metallic spin glasses are characterized by randomly distributed magnetic moments with so-called RKKY-interactions falling off as r^{-3} with distance [1]. A given spin has hence order $r^2 \mathrm{d}r$ couplings to spins a distance r away or, equivalently, $\mathrm{d}J/J^2$ couplings of strength J. Such a broad distribution of couplings is only poorly modelled by a Gaussian distribution usually used in theoretical descriptions of spin glasses. The Lévy spin glass, introduced in [2], is a mean-field spin glass model where the distribution of couplings has a power law tail with a diverging second moment. It gives a better description of many experimental glasses (metallic spin glasses, but also dipolar glasses) than the usual Sherrington-Kirkpatrick (SK) model [3]. It also provides a situation which is intermediate between the SK model and finite-connectivity mean-field spin glasses [4–7]. It is particularly relevant for the study of the importance of rare, but strong, coupling constants. In this respect, its understanding is an important step in the theory of spin glasses, as models with strong hierarchies of coupling strengths can be used to approach the finite-dimensional spin glass problem [8]. The problem has been around for long, but its physics has never been fully elucidated. The broad distribution of couplings presents both conceptual obstacles, and technical ones. In their pioneering work, Cizeau and Bouchaud [2] have computed the spin glass transition temperature, and argued that these rare and strong couplings can stabilize a replica-symmetric (RS) stable spin glass phase in the absence of external magnetic field. Here we revisit this problem using the RS cavity method. We show that replica symmetry is always broken in the spin glass phase, and we compute the de Almeida-Thouless (AT) line [9] giving the phase diagram as a function of temperature and magnetic field.

We consider an Ising spin glass with Hamiltonian

$$H(\{S_i\}) = -\frac{1}{2}\sum_{(i,j)} J_{ij} S_i S_j - h_{\text{ext}} \sum_i S_i, \qquad (1)$$

where the sum is over all pairs of spins $S_i = \pm 1$, $i = 1, \dots, N$ and h_{ext} denotes an external field. The couplings $J_{ij} = J_{ji}$ are independent, identically distributed random variables drawn from a distribution $P_\alpha(J) = N^{1/\alpha}\mathcal{P}(JN^{1/\alpha})$, where $\mathcal{P}(x)$ is a symmetric probability density with a power law tail which is that of a Lévy distribution with parameter $\alpha \in]1,2[$ [10]:

$$\mathcal{P}(x) \simeq_{|x|\to\infty} \frac{C}{|x|^{\alpha+1}}. \qquad (2)$$

As we shall see, the only important feature of $\mathcal{P}$ is this tail. We shall use specifically the function $\mathcal{P}(x) = \frac{\alpha}{2}\frac{1}{|x|^{\alpha+1}}\theta(|x|-1)$, where $\theta(x)$ denotes the Heaviside function. The scaling of the couplings with N ensures that the free energy corresponding to the Hamiltonian (1) is extensive [2,11].

The equilibrium thermodynamic properties of the system at temperature $1/\beta$ can be deduced from the

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probability distribution $P(h)$ of local fields h_i parameterizing the marginal distribution of spin variables by $P(S_i)=e^{\beta h_i S_i}/2\cosh(\beta h_i)$. Adding a new site $i=0$ with corresponding couplings J_{0i} to the system the new field h_0 is given by [12]

$$h_0 = h_{\text{ext}} + \sum_{i=1}^{N} u(h_i, J_{0i}), \tag{3}$$

where $u(h,J)=\text{atanh}(\tanh(\beta h)\tanh(\beta J))/\beta$.

This relation can be turned into a self-consistent equation for $P(h)$ by averaging over h_i and J_{0i}. Within the assumption of replica symmetry the h_i are independent and we find in the thermodynamic limit $N\to\infty$

$$\begin{aligned} P(h) &= \int\prod_i \mathrm{d}h_i P(h_i)\int\prod_i \mathrm{d}J_{0i} P_\alpha(J_{0i}) \\ &\quad\times\delta\left(h-h_{\text{ext}}-\sum_{i=1}^{N} u(h_i,J_{0i})\right) \\ &\to \int\frac{\mathrm{d}s}{2\pi}\exp\left[is(h-h_{\text{ext}})+\frac{\alpha}{2}\int\mathrm{d}h' P(h')\right. \\ &\quad\left.\times\int_{-\infty}^{\infty}\frac{\mathrm{d}J}{|J|^{\alpha+1}}\left(e^{-isu(h',J)}-1\right)\right]. \end{aligned} \tag{4}$$

For $h_{\text{ext}}=0$ this equation is equivalent to the one obtained in [11] using the replica method. Notice that the details of $\mathcal{P}(x)$ do not matter: the field distribution $P(h)$ depends only on the Lévy tail $C/|J|^{\alpha+1}$ of the distribution of couplings P_α, and not on the details of its regularization at small J. All the thermodynamic properties of the Levy spin glass only depend on the exponent α which characterizes the power law decay at large $|J|$, and the prefactor C (which in our case is chosen to be $C=\alpha/2$), which fixes the energy scale.

It is instructive to solve (4) numerically with a population dynamics [7] method. In order to do this, one should first realize that in the update equation (3) the main contribution is obtained from the relatively rare couplings which are finite in the large-N limit. Let us introduce a threshold ε and divide the couplings into strong ($|J_{ij}|>\varepsilon$) and weak ($|J_{ij}|\leqslant\varepsilon$) couplings. Equation (3) involves a sum over $\mathcal{O}(\varepsilon^{-\alpha})$ strong couplings, which is treated exactly, and a sum over $\mathcal{O}(N)$ weak ones, which is approximated by a Gaussian random variable z with zero mean and a variance determined self-consistently. The resulting population dynamics algorithm is given by [13]

$$\begin{aligned} h_j &= h_{\text{ext}} + \sum_{k=1}^{K} u(h_k, J_k) + z, \\ \overline{z^2} &= \alpha\int\mathrm{d}h P(h)\int_0^{\varepsilon}\frac{\mathrm{d}J}{J^{\alpha+1}}\,u^2(h,J), \end{aligned} \tag{5}$$

where K is a Poissonian with average $\varepsilon^{-\alpha}$. In this form the algorithm represents a noisy variant of the one used for locally tree-like graphs [7]. The most time-consuming step in its numerical implementation is the update of $\overline{z^2}$ each time a new field is generated in the population. In order to perform this step efficiently we first determine once for all an estimate of the function $h\to\int_0^\varepsilon\frac{\mathrm{d}J}{J^{\alpha+1}}\,u^2(h,J)$, based on a tabulation and interpolation procedure.

When $h_{\text{ext}}=0$, one easily finds that $P(h)=\delta(h)$ for $T>T_c(\alpha)$, and $P(h)$ becomes non-trivial at $T<T_c(\alpha)$. The spin glass transition temperature $T_c(\alpha)$ is independent of ε, it is given by

$$T_c(\alpha) = \left[\int_0^\infty \frac{\alpha\,\mathrm{d}x}{x^{\alpha+1}}\tanh^2 x\right]^{\frac{1}{\alpha}} \tag{6}$$

as found in [2,11].

The parameter ε plays a crucial role. The correct (within the RS approximation) $P(h)$ is obtained in the limit $\varepsilon\to 0$ whereas the limit $\varepsilon\to\infty$ amounts to approximating $P(h)$ by a Gaussian, as in [2]. For a given value of ε, there are on average $\varepsilon^{-\alpha}$ strong and $(N-\varepsilon^{-\alpha})$ weak bonds in eq. (3). In the large-N limit the total contribution of the weak bonds to the local field vanishes when $\varepsilon\to 0$. This becomes apparent from the fact that in this limit, $\overline{z^2}\simeq(\alpha/(2-\alpha))\varepsilon^{2-\alpha}\int\mathrm{d}hP(h)\tanh^2(\beta h)$ which also validates the exchange of the limits $\varepsilon\to 0$ and $N\to\infty$. The simplest procedure would hence be to neglect the weak bonds altogether by putting $z=0$ in (5). Although this would give the correct result when $\varepsilon\to 0$, the convergence of the procedure were much too slow for it to be of any practical use. Approximating the effect of weak bonds by a Gaussian random variable speeds up considerably the convergence of $P(h)$ towards its limiting $\varepsilon=0$ form. We show in fig. 1 the second moment $\langle h^2\rangle$ of $P(h)$ as a function of ε. The results obtained for $\varepsilon=0.2\ldots0.5$ are already very close to the exact value at $\varepsilon=0$. As a test we have also used the naive $z=0$ truncation for this $\alpha=1.1$ case. In order to obtain the same accuracy for $P(h)$ we had to go down to $\varepsilon=0.005$ entailing a factor 100 increase in computer time.

In zero external field, we have checked the result by a direct iteration of (4) using the fast Fourier transform. Figure 1 shows some examples of $P(h)$, together with the Gaussian form proposed in [2]. It is clear that $P(h)$ deviates from a Gaussian distribution. This can already be seen from (4): inserting a Gaussian $P(h')$ in the r.h.s. does not produce one in the l.h.s.

We now turn to the computation of the AT line, characterized by replica symmetry breaking (RSB). The RS cavity method described above is valid as long as the spin glass susceptibility, $\chi_{SG}=\sum_{i,j}\overline{(\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle)^2}/N$, is finite. The divergence of χ_{SG} signals the appearance of the spin glass phase. In order to compute this susceptibility, we use the truncated model where we keep only the strong bonds with $|J_{ij}|>\varepsilon$, while the weak bonds are neglected. Since the limit $\varepsilon\to 0$ will be performed analytically the truncated model is here an efficient approach. In the truncated model, the graph of interacting spins is a diluted Erdös-Renyí random graph: in the $N\to\infty$ limit

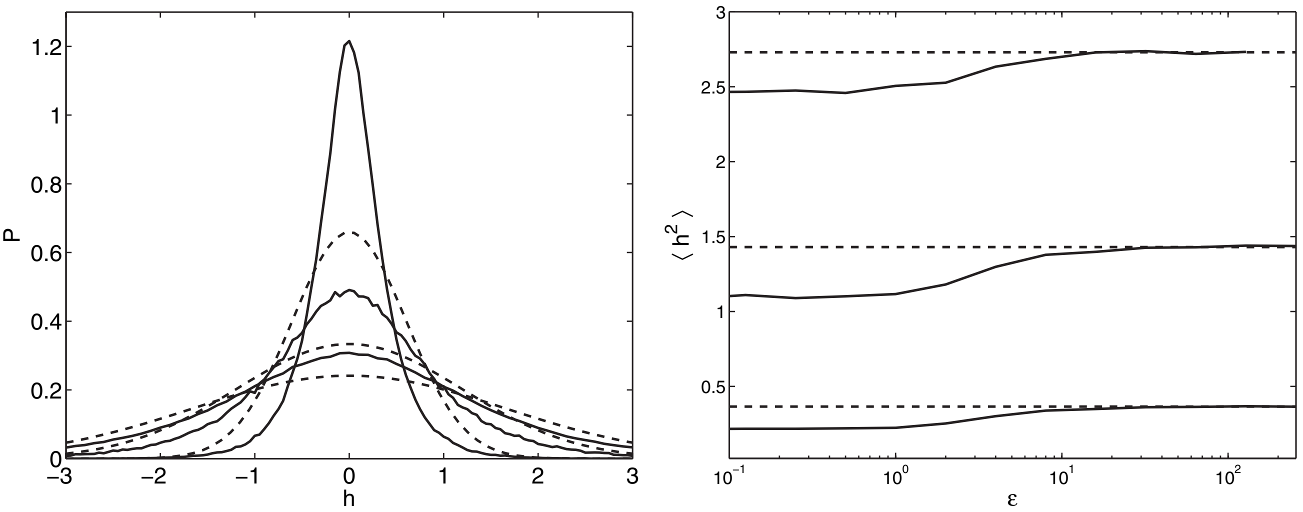

Fig. 1: Distribution of local fields $P(h)$, for $\alpha = 1.1$ and three different temperatures. The dotted lines are the corresponding results within the Gaussian approximation proposed in [2]. Left: $P(h)$ for $T = 0.9\,T_c$, $T = 0.6\,T_c$, and $T = 0.1\,T_c$ (center top to bottom) obtained from population dynamics with $\varepsilon = 0.3$. Right: the second moment of $P(h)$ for $T = 0.9\,T_c$, $T = 0.6\,T_c$, and $T = 0.1\,T_c$ (bottom to top), obtained with the population dynamics method as a function of the regularization parameter ε.

(taken before the $\varepsilon \to 0$ limit), the number of spins interacting with a given spin is a Poissonian random variable with mean $\varepsilon^{-\alpha}$. This graph is locally tree-like, in the sense that, if one looks at all the spins at distance $\leqslant r$ of a given spin S_i, their interaction graph is typically, in the large-N limit, a tree of depth r. This allows to compute χ_{SG} as [7,12]:

$$\chi_{SG} = \sum_{r=1}^{\infty} \varepsilon^{-\alpha r} C_2(r), \tag{7}$$

where $C_2(r)$ is the average square correlation, $\overline{(\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle)^2}$, between two sites i,j at distance r. As we will see, $C_2(r)$ decays exponentially with distance as $C_2(r) = Ae^{-r/\xi}$. We thus define the stability parameter $\lambda = \varepsilon^{-\alpha} e^{-1/\xi}$. This parameter is the rate of the geometric series (7) giving χ_{SG}. The spin glass phase transition is given by the condition $\lambda = 1$.

Because of the locally tree-like structure of the interaction graph in the truncated model, the computation of λ reduces to the study of a one-dimensional Lévy spin glass model, with energy given by

$$E = -\sum_{n=1}^{r-1} J_n S_n S_{n+1} - \sum_{n=1}^{r} h_n S_n, \tag{8}$$

where the couplings J_n are independent random variables drawn from the distribution $P_{\alpha,\varepsilon}(J) = \alpha\varepsilon^{\alpha}/(2|J|^{1+\alpha})\theta(|J| - \varepsilon)$, and h_n are independent random variables drawn from the distribution of cavity fields $P(h)$ determined above. $C_2(r)$ is the *average* square correlation $\overline{(\langle S_1 S_r\rangle - \langle S_1\rangle\langle S_r\rangle)^2}$, and one is interested in computing the decay rate $1/\xi = -\lim_{r\to\infty}\log(C_2(r))/r$. While this one-dimensional system looks simple, it requires some special care. The usual "population approach" used in finite-connectivity spin glasses [7,12,14] fails in the Lévy case, because the ratio between the average and the typical correlation diverges in the small ε limit: the well-known "non–self-averageness" of correlation functions [15] becomes crucial in this case. This fact is most easily seen in the case where the fields h_n are equal to zero. As we have seen, this happens when $h_{\text{ext}} = 0$ and $T > T_c(\alpha)$. The average correlation is $C_2(r) = (\int \mathrm{d}J P_{\alpha,\varepsilon}(J)\tanh^2(\beta J))^r$; in the limit where ε goes to 0 this gives $e^{-1/\xi} = \varepsilon^{\alpha}\int_0^{\infty}(\alpha \mathrm{d}J/J^{1+\alpha})\tanh^2(\beta J)$. Therefore the stability parameter is $\lambda = \int_0^{\infty}(\alpha\mathrm{d}J/J^{1+\alpha})\tanh^2(\beta J)$: the divergence of the spin glass susceptibility occurs exactly at the value T_c given by (6) where the distribution of local fields becomes non-trivial. The typical correlation is $\exp(r\int \mathrm{d}J P_{\alpha,\varepsilon}(J)\log(\tanh^2\beta J))$, it behaves as $\varepsilon^{2r} \ll \varepsilon^{\alpha r}$ in the small ε limit. This means that the average correlation $C_2(r)$ is totally dominated by rare realizations: its numerical estimate would require an average over $\mathcal{O}(1/\varepsilon^{(2-\alpha)r})$ samples.

In order to get around this problem, one must solve analytically the one-dimensional Lévy spin glass problem described in (8). This can be done either with the replica approach of [16], or using a cavity type approach. Both methods give the same result, the detailed computations will be given in [13]. Let us just describe in a nutshell the basic steps of the cavity approach. One first solves the one-dimensional spin glass model (8) using the cavity method. The solution is given in terms of some cavity fields g_n which satisfy the update equations $g_{n+1} = h_{n+1} + u(g_n, J_n)$. Then one studies the spin glass correlation through the response of g_n to a perturbation in g_1. Calling $\Delta_n = (\partial g_n/\partial g_1)^2$, linear response theory gives $\Delta_{n+1} = (\partial u(g_n, J_n)/\partial g_n)^2\Delta_n$. Let us denote by $P_n(g_n, \Delta_n)$ the joint probability distribution of g_n and

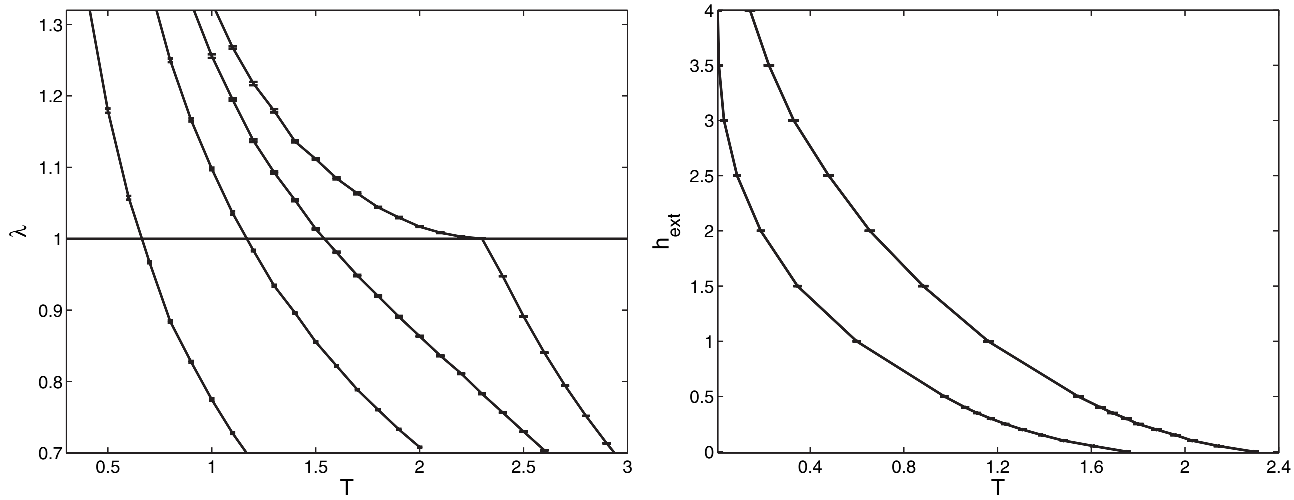

Fig. 2: Determination of the de Almeida-Thouless line. Left: stability parameter λ as a function of temperature for $\alpha = 1.5$ and $h_{\text{ext}} = 0$, 0.5, 1 and 2 (from right to left). From the intersection of the curves with the stability boundary $\lambda = 1$ the AT line is determined. Right: phase diagram of a Lévy spin glass with $\alpha = 1.5$ (right) and $\alpha = 1.1$ (left). Above the AT lines shown RS is stable, below it is unstable.

Δ_n, over realizations of the random variables $\{h_p\}$, $p \in \{1, n\}$, and $\{J_p\}$, $p \in \{1, n-1\}$. The update equations giving g_{n+1} and Δ_{n+1} in terms of g_n and Δ_n induce a mapping $P_{n+1} = F(P_n)$ for the joint probability distribution. In order to study this mapping, one can introduce the function $f_n(g_n) = \int \mathrm{d}\Delta_n \Delta_n P_n(g_n, \Delta_n)$. It satisfies the recursion relation

$$f_{n+1}(g_{n+1}) = \int \mathrm{d}g_n \int \mathrm{d}J_n P_{\alpha,\varepsilon}(J_n) \int \mathrm{d}h_{n+1} P(h_{n+1}) \times \left(\frac{\partial u(g_n, J_n)}{dg_n} \right)^2 f_n(g_n)\, \delta\left(g_{n+1} - [h_{n+1} + u(g_n, J_n)]\right). \tag{9}$$

In this equation one can safely take the $\varepsilon \to 0$ limit, as the function $(\partial u(g_n, J_n)/\partial g_n)^2$ behaves as J_n^2 at small J_n, cancelling the potential divergence of the Lévy distribution at small J_n. The resulting linear equation, $f_{n+1}(g_{n+1}) = \int \mathrm{d}g_n K(g_{n+1}, g_n) f_n(g_n)$, defines the transfer matrix operator $K(x, y)$. The correlation length ξ is given in terms of the largest eigenvalue ν of K by $\nu = e^{-1/\xi}$. The computation of ν is most easily done by changing from the right to the left eigenvalue equation. This gives the eigenvalue equation

$$\nu\phi(x) = \int \mathrm{d}J P_{\alpha,\varepsilon}(J) \int \mathrm{d}h P(h) \left(\frac{\partial u(x, J)}{\partial x} \right)^2 \times \phi(h + u(x, J)) = \int \mathrm{d}y\, K^T(x, y)\, \phi(y). \tag{10}$$

The largest eigenvalue of the linear operator K can be found numerically by iterating (10) $\phi_n(x) = \int \mathrm{d}y K^T(x, y) \phi_{n-1}(y)/Z_n$, starting from an arbitrary function $\phi_0(x)$. At each step the constant Z_n is computed by imposing a normalisation condition $\int \mathrm{d}x \phi_n(x) = 1$. After many iterations the function $\phi_n(x)$ converges to the eigenvector of K with the largest eigenvalue, and the normalisation converges to $\lim_{n\to\infty} Z_n = \nu = \exp(-1/\xi)$.

In order to find the AT line one must hence use the $P(h)$ distribution as determined above with (5) and then find the correlation length ξ of the one-dimensional problem using the ϕ_n iteration in (10). With this procedure the limit $\varepsilon \to 0$ is smooth, and this allows for a clean determination of the AT line, as shown in fig. 2.

The behaviour of the stability parameter λ can be studied analytically in zero external field close to the critical temperature. Writing $\tau = 1 - T/T_c(\alpha)$, one must compute the second and fourth moments of $P(h)$ up to order τ^2, and then expand the eigenvalue equation (10). One finds after some work $\lambda = 1 + (\alpha^2/3)\,(T_2(\alpha) + 2T_4(\alpha))/(T_2(\alpha) - T_4(\alpha))\,\tau^2 + O(\tau^3)$, where $T_n(\alpha) = \int_0^\infty \alpha \mathrm{d}x/x^{1+\alpha} \tanh^n x$. As the coefficient of τ^2 is positive for all $\alpha \in]1, 2[$, the RS solution is always unstable close to T_c, contrary to what was found with the Gaussian ansatz [2]. The same is obtained numerically in presence of an external field: we have not found any evidence for a stable RS spin glass phase, at all the values of α and h_{ext} that we have studied.

To summarize, we have shown how the Lévy spin glass problem can be studied naturally within the framework of diluted spin glasses, using a decomposition of the couplings into strong and weak ones. The resulting phase diagram is very similar to the one found in other mean-field spin glasses. In particular, the spin glass phase is never replica-symmetric. The large fluctuations due to the presence of rare strong couplings request the introduction of some rather sophisticated methods in order to compute the spin glass instability. These fluctuations are even more pronounced in the case $\alpha < 1$, not treated here, where the free energy ceases to be self-averaging. A natural question concerns the nature of the low-temperature spin glass phase. As we have seen the whole Lévy glass problem can be basically mapped to some kind of dilute spin glass problem, and it is likely that one will find the same phenomenology as in standard dilute spin glasses [7,17]: in those models the full RSB solution has not been found, but the one step RSB solution can be worked out in details and gives a very good approximation to the thermodynamic properties and ground-state energy. Applying similar methods to the Lévy glass is in principle straightforward, but in practice this computation requires developing some sophisticated techniques for dealing with the broad distribution of couplings, as we have seen in the computations of the AT line. This is left for future work [13].

While we were writing up this work, a preprint by Neri *et al.*, now published in [18] has appeared where similar issues were addressed. This paper uses the same kind of decomposition into strong and weak couplings as we do. It is complementary to ours in that it studies the phase diagram of a Lévy spin glass without external field, but with a bias in the coupling distribution, using a two-replica method which is different from our computation of the spin glass susceptibility.

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REFERENCES

[1] Binder K. and Young A. P., *Rev. Mod. Phys.*, **58** (1986) 801.
[2] Cizeau P. and Bouchaud J.-P., *J. Phys. A*, **26** (1993) L187.
[3] Sherrington D. and Kirkpatrick S., *Phys. Rev. Lett.*, **35** (1975) 1972.
[4] Viana L. and Bray A. J., *J. Phys. C*, **18** (1985) 3037.
[5] Kanter I. and Sompolinski H., *Phys. Rev. Lett.*, **58** (1987) 164.
[6] Mézard M. and Parisi G., *Europhys. Lett.*, **3** (1987) 1067.
[7] Mézard M. and Parisi G., *Eur. Phys. J. B*, **20** (2001) 217.
[8] Newman C. M. and Stein D. L., *Phys. Rev. Lett.*, **72** (1994) 2286.
[9] de Almeida J. R. L. and Thouless D. J., *J. Phys. A*, **11** (1978) 983.
[10] Gnedenko B. V. and Kolmogorov A. N., *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, Mass.) 1954.
[11] Janzen K., Hartmann A. K. and Engel A., *J. Stat. Mech.* (2008) P04006.
[12] Mézard M. and Montanari A., *Information, Physics and Computation* (Oxford University Press, Oxford) 2009.
[13] Janzen K., Engel A. and Mézard M., in preparation.
[14] Jörg T., Katzgraber H. and Krzakala F., *Phys. Rev. Lett.*, **100** (2008) 197202.
[15] Derrida B. and Hilhorst H. J., *J. Phys. C*, **14** (1981) L539.
[16] Weigt M. and Monasson R., *Europhys. Lett.*, **36** (1996) 209.
[17] Mézard M. and Parisi G., *J. Stat. Phys.*, **111** (2003) 1.
[18] Neri I., Metz F. L. and Bollé D., *J. Stat. Mech.* (2010) P01010.