Disorder-Driven Quantum Phase Transitions in Superconductors and Magnets

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(Received 14 September 2009; published 14 July 2010)

We develop an analytical theory, based on the quantum cavity method, describing the quantum phase transitions in low-temperature, strongly disordered ferromagnets and superconductors. At variance with the usual quantum critical points, we find a phase diagram with two critical points separating three phases. When the disorder increases, the systems goes from the ordered phase to an intermediate disordered phase characterized by activated transport and then to a second disordered phase where no transport is possible. Both the ordered and disordered phases exhibit strong inhomogeneity of their basic properties typical of glassy physics.

DOI: 10.1103/PhysRevLett.105.037001 PACS numbers: 74.81.−g, 71.30.+h, 74.25.F−, 75.47.−m

The zero-temperature quantum phase transitions and their quantum critical points have been rather well understood in translationally invariant systems [1]. Much less is known about disordered systems where the transition is driven by the competition between strong disorder and interactions.

Motivated by experiments on disordered ferromagnets and superconducting films [2], we formulate and solve a theoretical model of a general disorder-driven transition in the quantum (low-temperature) regime. Recent work has shown that the strong disorder can be at the origin of new phases in which all or some excitations are localized in space and have infinite lifetime and thus cannot contribute to any transport [3]. Our model shows that the phase transition where the long-range order disappears has many other features that distinguish it from conventional quantum critical points.

By solving the model on a Bethe lattice, we find that the zero-temperature quantum phase transition happens in two steps upon increase of the disorder. In the first phase formed when order is destroyed, only low-energy local excitations have infinitely long lifetime while high energy ones can decay; in the second phase all excitations acquire infinite lifetime. The energy separating the two types of excitations in the first phase becomes zero at the quantum critical point and shoots up at the second phase boundary. The first transition is characterized by wide distributions of the order parameter (in the ordered phase) and of the relaxation rates (in the first disordered phase). Their typical values depend exponentially on the parameters of the model (interaction strength and disorder), in contrast to the power laws of usual quantum phase transitions. These predictions differ from those of the mean-field theory that gives small but nonzero value of the critical temperature for any disorder. Because the Bethe lattice is in many respects similar to a high dimensional lattice where mean-field theory is exact, the very existence of such phase transition was unexpected.

We argue that the most plausible mechanism for the superconductor-insulator transition in homogeneous disordered films of InO, TiN, or Be [4] is a competition between pair hopping and random pair energies on different sites, as suggested in a seminal paper of Ma and Lee [5]. In the vicinity of this quantum critical point, the tunneling spectroscopy shows a well defined gap at all points. However, the coherence peaks expected for a BCS superconductor appear at some locations and disappear at others [6]. These results are expected [7,8] if the disorder does not affect local pairing of electrons but prevents the formation of a coherent state of these pairs. In contrast, the Coulomb-interaction-based mechanism would decrease both the gap and the superconducting coherence, so one can exclude it as driving force of the transition.

This Letter studies the Ising model in a random transverse field beyond the simple mean-field analysis. This model is directly applicable to disordered ferromagnets. For the lattice characterized by connectivity Z our formalism keeps the leading-order terms in $1/\ln Z$ but neglects terms that are of the order of $1/Z$; in this approximation the results obtained in the ferromagnet coincide with those of the Ma-Lee model. When applied to the superconductor-insulator transition these results reproduce correctly the most important features of experiments on disordered films: direct superconductor-insulator transition, activated behavior close to the quantum critical point in the insulating phase, strong dependence of the activation energy near the quantum critical point and huge order parameter variations from site to site in the superconducting phase.

The model is described by the Hamiltonian

$$H = -\left(\sum_i \xi_i \sigma_i^z + \frac{g}{Z-1}\sum_{(ij)} \sigma_i^x \sigma_j^x\right), \qquad (1)$$

where ξ_j's are quenched random variables drawn from a probability $P(\xi)$. The second sum runs over the edge of a random graph with exactly Z neighbors for each site. With a redefinition of the meaning of spin up and down, one can

take the site energies ξ_j to be all positive. We shall assume that the ξ are uniformly distributed in the interval $[0, 1/\nu]$, and choose $\nu = 1$ as energy scale.

The Ma-Lee model assumes well paired electrons that hop from one localized single electron state to another. Its low-energy physics is described by the spin Hamiltonian [5,7,9]:

$$H = -\Big(\sum_i \xi_i \sigma_i^z + \sum_{(ij)} M_{ij}(\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)\Big). \quad (2)$$

Here the state with $\sigma_i^z = \pm 1$ corresponds to a local level occupied or unoccupied by a Cooper pair; M_{ij} is the pair hopping amplitude between sites i and j. These hopping amplitudes couple a typical local level to a large number of neighbors, $Z \gg 1$. Although the model (2) has Goldstone modes, which are absent in (1), these modes produce effects small in $1/Z$ that do not affect qualitatively the phase diagram; furthermore, the effect of these modes in realistic three-dimensional systems is small at low temperatures [7] because of the small value of the Ginzburg parameter that controls the thermodynamic fluctuations even in the vicinity of the transition. Qualitatively, this is due to the fact that transition is driven by short scale phenomena. In the language of (1) and (2), in the superconducting phase a spontaneous magnetization appears in the x direction; in the insulating phase the spins point parallel to the z axis.

In a simple mean-field (SMF) approach to this problem, H is replaced by $H_{\rm MF} = \sum_i (-\xi_i \sigma_i^z - B\sigma_i^x)$ and B is determined self-consistently as $B = (g/Z)\sum_j \langle \sigma_j^x \rangle$. At temperature $T = 1/\beta$, this gives $B = (g/Z)\sum_j [B/\sqrt{\xi_j^2 + B^2}] \tanh(\beta\sqrt{\xi_j^2 + B^2})$, which self-averages in the large Z limit. The SMF thus predicts a phase transition from insulator to superconductor at the critical value of the hopping $g_c^{\rm SMF} = (\int d\xi P(\xi) \times \tanh(\beta\xi)/\xi)^{-1}$. As $P(0) > 0$, $g_c^{\rm SMF} \to 0$ when $T \to 0$.

While these SMF predictions are correct at $Z = \infty$, they are qualitatively wrong at low temperature in finite connectivity systems, in particular $g_c^{\rm SMF}(T \to 0) \neq 0$, as first argued in [10]. We now turn to a more refined approximation, valid for finite $Z \gg 1$, which is the basis for our results. We use a quantum version of the cavity method [11] which becomes exact for spins on a Bethe lattice of connectivity Z. In this method, one studies the properties of a spin j in the cavity graph where one of its neighbors has been deleted, assuming that the $K = Z - 1$ remaining neighbors are uncorrelated. The full quantum cavity method [12,13] involves a complicated mapping of spin trajectories in imaginary time. Here we use a simplified version which projects onto the trajectories generated by a local Hamiltonian (these are the most important ones at large Z). The system of spin j and its K neighbors is described by the Hamiltonian

$$H_j^{\rm cav} = -\xi_j \sigma_j^z - \sum_{k=1}^{K} \Big(\xi_k \sigma_k^z + B_k \sigma_k^x + \frac{g}{K} \sigma_j^x \sigma_k^x\Big), \quad (3)$$

where B_k is the local "cavity" field on spin k due to the rest of the spins (in absence of j). By solving the problem of Z Ising spins in (3), one can compute the induced magnetization of j, $\langle \sigma_j^x \rangle$, which is by definition equal to $[B_j/\sqrt{\xi_j^2 + B_j^2}] \tanh(\beta B_j/\sqrt{\xi_j^2 + B_j^2})$. We thus get a mapping that gives the new cavity field B_j in terms of the K fields B_k on the neighboring spins [14]. This cavity mapping induces a self-consistent equation for the distribution of the B fields [11]. This mapping cannot be written explicitely as it involves the diagonalization of the Z-spins cavity Hamiltonian (3), but it can be studied numerically for moderate Z. A good analytic approximation can be obtained through a mean-field study of (3), which gives the explicit mapping

$$B_j = \frac{g}{K} \sum_{k=1}^{K} \frac{B_k}{\sqrt{B_k^2 + \xi_k^2}} \tanh\beta\sqrt{B_k^2 + \xi_k^2}. \quad (4)$$

This mean-field approximation of the cavity mapping neglects some level repulsion effects, but by comparing its results with a numerical study of the cavity mapping, we have checked that it reproduces the qualitative features of the phase diagram and becomes quantitatively correct in the limit of $K \gg 1$.

In order to understand the mapping (4), let us imagine that we iterate it R times on a Bethe lattice. When the number of spins is large, the corresponding graph is a rooted tree with branching factor K at each node and depth R. The field B_0 at the root is a function of the K^R fields on the boundary. In order to detect spontaneous ordering, we compute B_0 in linear response to infinitesimal fields $B_i = B \ll 1$ on the boundary spins. This is given by

$$B_0/B = \Xi \equiv \sum_P \prod_{n \in P} \Big[\frac{g}{K} \frac{\tanh(\beta\xi_n)}{\xi_n}\Big], \quad (5)$$

where the sum is over all paths going from the root to the boundary, and the product $\prod_{n \in P}$ is over all sites along the path P. The response Ξ is nothing but the partition function for a directed polymer (DP) on a tree, where the energy of each site is $e^{-E_n} = (g/K)(\tanh(\beta\xi_n)/\xi_n)$ and the temperature has been set equal to one. The solution of this problem, found in [15], can be expressed in terms of the convex function $f(x) = (1/x)\log[K \int_0^1 d\xi (\tanh(\beta\xi)/\xi)^x]$, which is minimal at a value $x = x_c$. In the large R limit, there exist two phases for the DP problem: (i) "Self-averaging" (SA) phase: If $x_c > 1$, then $(1/R) \times \log\Xi = f(1) + \log(g/K)$. The ordered phase appears at $g_c = Ke^{-f(1)} = g_c^{\rm SMF}$. (ii) "Glassy phase" (GL) phase: If $x_c < 1$, then $(1/R)\log\Xi = f(x_c) + \log(g/K)$. The ordered phase appears at $g_c = Ke^{-f(x_c))} > g_c^{\rm SMF}$. These two regimes of the DP problem are qualitatively very different.

The SA regime is the high "temperature" phase of the polymer, where the measure on paths defined in (5) is more or less evenly distributed among all paths. The low-temperature GL regime is a glass phase where the measure condensates onto a small number of paths. An order parameter which distinguishes between these phases is the participation ratio $Y = \sum_P w_P^2$, where w_P is the relative weight of path P in the measure (5). In the replica formalism the SA phase is replica symmetric, Ξ is self-averaging, and $Y = 0$; the GL phase is a one-step replica-symmetry-breaking (RSB) glass phase, the value of Y is finite and non-self-averaging (it depends on the explicit realization of the ξ's even in the thermodynamic limit), and its average is given by $1 - x_c$ [16]. This glass transition, and the nature of the GL phase, are identical to the ones found in the random energy model [17,18].

Using these DP results one gets the phase diagram of the spin systems shown in the right pane of Fig. 1. At any temperature, there is a nonzero critical value of the coupling, $g_c(T)$, separating an ordered, superconducting phase with spontaneous x magnetization at $g > g_c(T)$ from a normal disordered, insulating phase with zero magnetization at $g < g_c(T)$. Within each phase there are two regimes of temperature, SA and GL. As is clear from our susceptibility analysis, the glass transition of the DP affects the propagation of a static perturbation in the spin system. In the disordered SA phase, the total effect of the perturbation decreases when R increases, and propagates evenly: the average value of the susceptibility coincides with its typical value. In the disordered GL phase, the total perturbation also decays, but it condenses on a finite number of paths. Consequently, the susceptibility is non-self-averaging, similarly to what is found in one dimension [19]. Rare paths are important in the whole glassy phase where the susceptibility distribution has a power law tail. The power tail leads to a divergence of the higher moments of the susceptibility even for $g > g_c^{\mathrm{SMF}}$, signalling the appearance of a Griffith phase in this regime as shown in Fig. 1, in agreement with numerical work on the 2D model [20]. When $g_c^{\mathrm{SMF}} < g < g_c$ the typical susceptibility is finite but the average susceptibility diverges. In the ordered phase, the perturbation propagates to infinity, again with very different patterns in the SA and GL phases. The SMF gets the correct result of the SA regimes, but completely misses the low-temperature physics of condensed correlation paths.

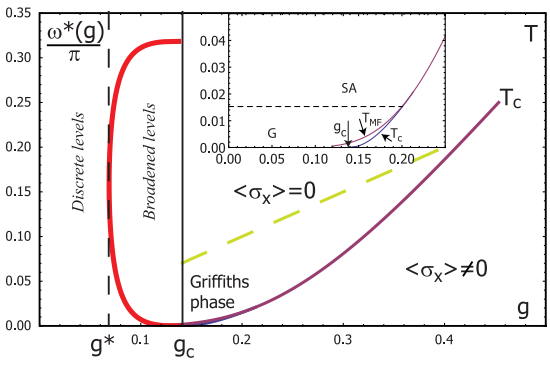

FIG. 1 (color online). Phase diagram of the spin system for $K = 2$. The right pane shows the critical line $T_c(g)$ (full lines) and the boundary of the Griffiths phase (dashed line). The left pane shows the critical energy that separates the states with zero width from those with a finite width. The inset shows the low-temperature region which emphasizes the difference between mean-field and Bethe lattice solutions.

The RSB transition also strongly affects the scaling of the field in the ordered phase, for $g \simeq g_c$ which is fully characterized by the distribution of fields $P(B)$ induced by (4). An expansion of its Laplace transform shows that, in the GL phase, $P(B)$ decays at large B as $P(B) \simeq C/B^{1+x_c}$. This distribution has a diverging mean, dominated by rare fluctuations. The analysis of the self-consistent equation for $P(B)$ shows that the geometric mean of the field behaves as $B_{\mathrm{typ}} \simeq A \exp[-g_B/(g - g_c)]$.

In the disordered phase the average value of the transverse field is zero, but its quantum fluctuations can become important and lead to a broadening of the local levels that, in the absence of g, correspond to $\sigma_i^z = \pm 1$. We first study this effect at $T = 0$. The level broadening (LB) means that local excitations of energy $\omega \simeq 2\xi_i$ decay. This is generically impossible in finite systems because of energy conservation. Therefore the onset of LB, which is associated to the possibility to transport energy, is a phase transition phenomenon which appears only in infinite systems. We can study this transition using the same type of approach as for the static phase diagram. We consider a central 'root' spin in a Bethe lattice of depth R, and assume that the system is very weakly coupled to the environment through its boundary spins. This coupling is described by adding to the Hamiltonian (1) the boundary term $H_{\mathrm{env}} = -\sum_{i\in B} \sigma_i^x B_i(t)$ where the sum is over the K^R boundary spins, and the $B_i(t)$ are independent *dynamical* random fields generated by the environment, characterized by a response function $G(\omega)\delta_{il} = \int dt \langle B_i(t) B_l(0)\rangle e^{-i\omega t}$.

In the leading order in g/K the relaxation rate of the root spin follows from the Fermi golden rule:

$$\Gamma_0(\omega) = \mathrm{Im} G(\omega) \sum_P \prod_{n\in P} \left[\frac{2g/K}{\omega - 2\xi_k}\right]^2. \qquad (6)$$

This perturbative equation is valid when all fractions inside the product remain small, and the relaxation rate of each spin is very small. Thus it is self-consistent if $\Gamma_0 \to 0$, which is enough to locate the LB transition.

Eqn. (6) is similar to (5) and can be studied with the same method. The typical value of the central spin width, $\Gamma_0(\omega)$, is controlled by

$$f_\Gamma(\omega) = \frac{1}{R} \int_0^1 \prod_n d\xi_n \ln\left\{\sum_P \prod_{n\in P} \left[\frac{2}{\omega - 2\xi_n}\right]^2\right\}.$$

$\Gamma_0(\omega)$ decreases away from the boundary and goes to zero

if and only if $f_\Gamma(\omega) + 2\ln(g/K) < 0$. This LB transition line can be obtained from the computation of f_Γ with the mapping to a DP, which gives the following results. At $\omega = 0$, in the whole disordered regime $g < g_c$, $f_\Gamma(0) + 2\ln(g/K) \leq 0$, and therefore the spontaneous level-width $\lim_{G\to 0}\lim_{R\to\infty}\Gamma_0$ vanishes. The LB transition appears exactly at the critical point $g = g_c$ where order parameter develops. As one might expect, $f_\Gamma(\omega)$ decreases with ω, it is minimal at $\omega = 1/2$ (which corresponds to the center of the band in our notations), see Fig. 1. At $g < g^* = Ke^{f_\Gamma(1/2)/2}$ the relaxation rate is zero for all states, this is the ''superinsulator'' regime of [3]. In the intermediate regime $g^* < g < g_c$ the states in the middle of the band have finite width, they are separated from the zero-width states by a critical energy $\omega^*(g)$ similar to the mobility edge of the noninteracting problem. Similar to the order parameter in the ordered phase the excitations reside on a small subset of all sites; this subset becomes nearly one-dimensional as $\omega \to \omega^*$.

We now discuss the low-temperature properties of the relaxation, neglecting phonons. An important ingredient is the level-width $\Gamma(\omega)$ when $\omega > \omega^*(g)$, which we have found [2] to behave as: $\Gamma_{\rm typ}(\omega) \simeq \Gamma^* \exp[-\omega_0(g)/(\omega - \omega^*(g))]$. In the intermediate regime $g^* < g < g_c$ the existence of some mobile excitations with frequencies above $\omega^*(g)$ provides a mechanism for a small broadening of the very low-energy levels, which can be estimated as follows. A mobile excitation with energy E appears with an Arrhenius rate, giving a width $\exp(-\omega_0/(E-\omega^*) - E/T)$. The dominant contribution comes from excitations with energies $E = \omega^*(g) + \sqrt{\omega_0 T}$, and results in the temperature dependence $\Gamma \sim \exp(-2\sqrt{\omega_0/T} - \omega^*(g,T)/T)$ that shows a crossover between a square root and activated (or even faster) behavior as one goes away from the critical point. Similar to the order parameter in the ordered phase, the wave functions of the mobile excitations form a very sparse tree near the critical frequency $\omega^*(g)$.

In conclusion, we have found that the Ising model in transverse field (and the equivalent Ma-Lee model) on the Bethe lattice shows a series of two zero-temperature transitions between a phase with no relaxation, a phase with a slow relaxation and an ordered phase. The low-temperature phases are very strongly nonuniform: both the order parameter formation and the spin relaxation are controlled by rare interaction paths containing a very small number of spins. When applied to the superconductor-insulator transition our results imply the existence of both weak and strong insulators. At the critical point the relaxation rate varies as $\exp(1/\sqrt{T})$ but crosses over to activated at lower g and low T, in the strong insulator the relaxation is completely suppressed. Of course, some physical effects neglected in our model would lead to a very slow relaxation even in the strong insulator.

We acknowledge very useful discussions with M. Feigel'man and with M. Mueller, who has obtained related results from phenomenological considerations [22]. This research was supported by Triangle de la physique 2007-36, ANR-06-BLAN-0218, ECS-0608842, ARO 56446-PH-QC, and DARPA HR0011-09-1-0009.

Recent data [21] confirm the prediction of the anomalously broad distribution of the order parameter close to the quantum superconductor-insulator transition.

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