

# On the Feasibility of Portfolio Optimization under Expected Shortfall

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We address the problem of portfolio optimization under the simplest coherent risk measure, i.e. the expected shortfall. As it is well known, one can map this problem into a linear programming setting. For some values of the external parameters, when the available time series is too short, the portfolio optimization is ill posed because it leads to unbounded positions, infinitely short on some assets and infinitely long on some others. As first observed by Kondor and coworkers, this phenomenon is actually a phase transition. We investigate the nature of this transition by means of a replica approach.

## I. INTRODUCTION

Among the several risk measures existing in the context of portfolio optimization, expected shortfall (ES) has certainly gained increasing popularity in recent years. In several practical applications, ES is starting to replace the classical Value-at-Risk (VaR). There are a number of reasons for this. For a given threshold probability  $\beta$ , the VaR is defined so that with probability  $\beta$  the loss will be smaller than VaR. This definition only gives the minimum loss one can reasonably expect but does not tell anything about the typical value of that loss that can be measured by the *conditional* value-at-risk (CVaR, which is the same as ES for continuous distributions that we consider here [13]). We will be more precise on these definitions below. The point we want to stress here is that the VaR measure, lacking the mandatory properties of subadditivity and convexity, is not *coherent* [1]. This means that summing VaR's of individual portfolios will not necessarily produce an upper bound for the VaR of the combined portfolio, thus contradicting the holy principle of diversification in finance. A nice practical example of the inconsistency of VaR in credit portfolio management is reported in Ref. 2. On the other hand, it has been shown [3] that ES is a coherent measure with interesting properties [4]. Moreover, the optimization of ES can be reduced to linear programming [5] (which allows for a fast implementation) and leads to a good estimate for the VaR as a byproduct of the minimization process. To summarize, the intuitive and simple character, together with the mathematical properties (coherence) and the fast algorithmic implementation (linear programming), are the main reasons behind the growing importance of ES as a risk measure.

In this paper, we will focus on the feasibility of the portfolio optimization problem under the ES measure of risk. The control parameters of this problem are (i) the imposed threshold in probability,  $\beta$ , and (ii) the ratio  $N/T$  between the number  $N$  of financial assets making up the portfolio and the time series length  $T$  used to sample the probability distribution of returns. (It is curious that, albeit trivial, the scaling in  $N/T$  had not been explicitly pointed out before [6].) It has been discovered in [7] that, for certain values of these parameters, the optimization problem does not have a finite solution because, even if convex, it is not bounded from below. Extended numerical simulations allowed these authors to determine the feasibility map of the problem. Here, in order to better understand the root of the problem and to study the transition from a feasible regime to an unfeasible one (corresponding to an ill-posed minimization problem) we address the same problem from an analytical point of view.

The paper is organized as follows. In Section II we briefly recall the basic definitions of  $\beta$ -VaR and  $\beta$ -CVaR and we show how the portfolio optimization problem can be reduced to linear programming. We introduce a “cost function” to be minimized under linear constraints and we discuss the rationale for a statistical mechanics approach. In Section III we solve the problem of optimizing large portfolios under ES using the replica approach. Our results and the comparison with numerics are reported in Section IV, and our conclusions are summarized in Section V.

## II. THE OPTIMIZATION PROBLEM

We consider a portfolio of  $N$  financial instruments  $\mathbf{w} = \{w_1, \dots, w_N\}$ , where  $w_i$  is the position of asset  $i$ . The global budget constraint fixes the sum of these numbers: we impose for example

$$\sum_{i=1}^N w_i = N. \quad (1)$$

We do not stipulate any constraint on short selling, so that  $w_i$  can be any negative or positive number. This is, of course, unrealistic for liquidity reasons, but considering this case allows us to show up the essence of the phenomenon.

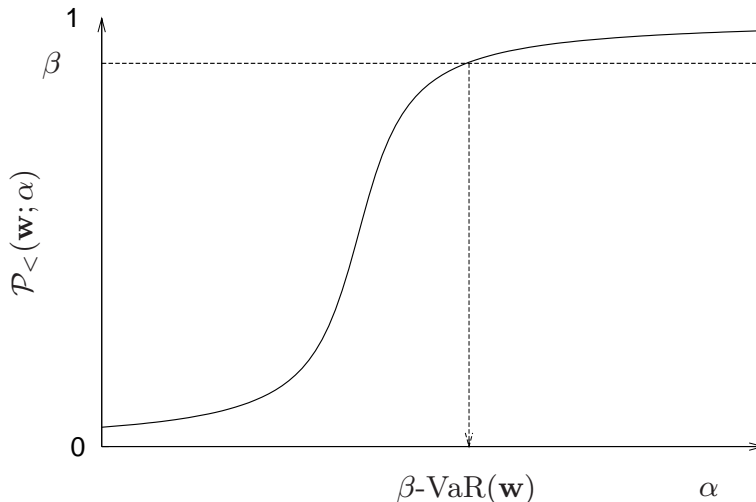


FIG. 1: Schematic representation of the VaR measure of risk.  $\mathcal{P}_{<}(\mathbf{w})$  is the probability of a loss associated to the portfolio  $\mathbf{w}$  being smaller than  $\alpha$ . The conditional VaR  $\beta$ -CVaR (or ES) is the average loss when this is constrained to be greater than the  $\beta$ -VaR.

If we imposed a constraint that would render the domain of the  $w_i$  bounded (such as a ban on short selling), this would evidently prevent the weights from diverging, but a vestige of the transition would still remain in the form of large, though finite, fluctuations of the weights, and in a large number of them sticking to the “walls” of the domain.

We denote the returns on the assets by  $\mathbf{x} = \{x_1, x_2, \dots, x_N\}$ , and we will assume an underlying normal distribution  $p(\mathbf{x}) \sim \prod_i \exp(-Nx_i^2/2)$ . The corresponding loss is  $\ell(\mathbf{w}|\mathbf{x}) = -\sum_{i=1}^N w_i x_i$ , and the probability of that loss being smaller than a given threshold  $\alpha$  is

$$\mathcal{P}_{<}(\mathbf{w}, \alpha) = \int d\mathbf{x} p(\mathbf{x}) \theta(\alpha - \ell(\mathbf{w}|\mathbf{x})) , \quad (2)$$

where  $\theta(\cdot)$  is the Heaviside step function, equal to 1 if its argument is positive and 0 otherwise. The  $\beta$ -VaR of this portfolio is formally defined by

$$\beta\text{-VaR}(\mathbf{w}) = \min\{\alpha : \mathcal{P}_{<}(\mathbf{w}, \alpha) \geq \beta\} , \quad (3)$$

(see Fig. 1), while the CVaR (or ES, in this case) associated with the same portfolio is the average loss on the tail of the distribution,

$$\beta\text{-CVaR}(\mathbf{w}) = \frac{\int d\mathbf{x} p(\mathbf{x}) \ell(\mathbf{w}|\mathbf{x}) \theta(\ell(\mathbf{w}|\mathbf{x}) - \beta\text{-VaR}(\mathbf{w}))}{\int d\mathbf{x} p(\mathbf{x}) \theta(\ell(\mathbf{w}|\mathbf{x}) - \beta\text{-VaR}(\mathbf{w}))} = \frac{1}{1-\beta} \int d\mathbf{x} p(\mathbf{x}) \ell(\mathbf{w}|\mathbf{x}) \theta(\ell(\mathbf{w}|\mathbf{x}) - \beta\text{-VaR}(\mathbf{w})) . \quad (4)$$

The threshold  $\beta$  then represents a confidence level. In practice, the typical values of  $\beta$  which one considers are  $\beta = 0.90, 0.95$ , and  $0.99$ , but we will address the problem for any  $\beta \in [0, 1]$ . What is usually called “exceedance probability” in some previous literature would correspond here to  $(1 - \beta)$ .

As mentioned in the introduction, the ES measure can be obtained from a variational principle [5]. The minimization of a properly chosen objective function leads directly to (4):

$$\beta\text{-CVaR}(\mathbf{w}) = \min_v F_\beta(\mathbf{w}, v) , \quad (5)$$

$$F_\beta(\mathbf{w}, v) \equiv v + (1 - \beta)^{-1} \int d\mathbf{x} p(\mathbf{x}) [\ell(\mathbf{w}|\mathbf{x}) - v]^+ . \quad (6)$$

Here,  $[a]^+ \equiv (a + |a|)/2$ . The external parameter  $v$  over which one has to minimize is claimed to be relevant in itself [5], since its optimal value may represent a good estimate for the actual value-at-risk of the portfolio. We will

come back to this point as we discuss our results. We stress here that minimizing (6) over  $\mathbf{w}$  and  $v$  is equivalent to optimizing (4) over the portfolio vectors  $\mathbf{w}$ .

Of course, in practical cases the probability distribution of the loss is not known and must be inferred from the past data. In other words, we need an “in-sample” estimate of the integral in (6), which would turn a well posed (but useless) optimization problem into a practical approach. We thus approximate the integral by sampling the probability distributions of returns. For a given time series  $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(T)}$ , our objective function becomes simply

$$\hat{F}_\beta(\mathbf{w}, v) = v + \frac{1}{(1-\beta)T} \sum_{\tau=1}^T [\ell(\mathbf{w}|\mathbf{x}^{(\tau)}) - v]^+ = v + \frac{1}{(1-\beta)T} \sum_{\tau=1}^T \left[ -v - \sum_{i=1}^N w_i x_{i\tau} \right]^+, \quad (7)$$

where we denote by  $x_{i\tau}$  the return of asset  $i$  at time  $\tau$ . Optimizing this risk measure is the same as the following linear programming problem:

- given one data sample, i.e. a matrix  $x_{i\tau}$ ,  $i = 1, \dots, N$ ,  $\tau = 1, \dots, T$ ,
- minimize the *cost function*

$$E_\beta[v, \{w_i\}, \{u_\tau\}; \{x_{i\tau}\}] = (1-\beta)Tv + \sum_{\tau=1}^T u_\tau, \quad (8)$$

- over the  $(N+T+1)$  variables  $\mathbf{Y} \equiv \{w_1, \dots, w_N, u_1, \dots, u_T, v\}$ ,
- under the  $(2T+1)$  constraints

$$u_\tau \geq 0, \quad u_\tau + v + \sum_{i=1}^N x_{i\tau} w_i \geq 0 \quad \forall \tau, \quad \text{and} \quad \sum_{i=1}^N w_i = N. \quad (9)$$

Since we allow short positions, not all the  $w_i$  are positive, which makes this problem different from standard linear programming. To keep the problem tractable, we impose the condition that  $w_i \geq -W$ , where  $W$  is a very large cutoff, and the optimization problem will be said to be ill-defined if its solution does not converge to a finite limit when  $W \rightarrow \infty$ . It is now clear why constraining all the  $w_i$  to be non-negative would eliminate the feasibility problem: a finite solution will always exist because the weights are by definition bounded, the worst case being an optimal portfolio with only one non-zero weight taking care of the total budget. The control parameters that govern the problem are the threshold  $\beta$  and the ratio  $N/T$  of assets to data points. The resulting “phase diagram” is then a line in the  $\beta$ - $N/T$  plane separating a region in which, with high probability, the minimization problem is not bounded and thus does not admit a finite solution, and another region in which a finite solution exists. These statements are non-deterministic because of the intrinsic probabilistic nature of the returns. We will address this minimization problem in the non-trivial limit where  $T \rightarrow \infty$ ,  $N \rightarrow \infty$ , while  $N/T$  stays finite. In this “thermodynamic” limit, we shall assume that extensive quantities (like the average loss of the optimal portfolio, i.e. the minimum cost function) do not fluctuate, namely that their probability distribution is concentrated around the mean value. This “self-averaging” property has been proven for a wide range of similar statistical mechanics models [8]. Then, we will be interested in the average value of the min of (8) over the distribution of returns. Given the similarity of portfolio optimization with the statistical physics of disordered systems, this problem can be addressed analytically by means of a replica approach [9].

### III. THE REPLICA APPROACH

We consider one given sample, i.e. a given history of returns  $x_{i\tau}$  drawn from the distribution

$$p(\{x_{i\tau}\}) \sim \prod_{i\tau} e^{-Nx_{i\tau}^2/2}. \quad (10)$$

In order to compute the minimal cost, we introduce the partition function at inverse temperature  $\gamma$ . Recalling that  $\mathbf{Y}$  is the set of all variables, the partition function at inverse temperature  $\gamma$  is defined as

$$Z_\gamma[\{x_{i\tau}\}] = \int_V d\mathbf{Y} \exp \left[ -\gamma E_\beta[\mathbf{Y}; \{x_{i\tau}\}] \right] \quad (11)$$

where  $V$  is the convex polytope defined by (9). The intensive minimal cost corresponding to this sample is then computed as

$$\varepsilon[\{x_{i\tau}\}] = \lim_{N \rightarrow \infty} \frac{\min E[\{x_{i\tau}\}]}{N} = \lim_{N \rightarrow \infty} \lim_{\gamma \rightarrow \infty} \frac{-1}{N\gamma} \log Z_\gamma[\{x_{i\tau}\}] . \quad (12)$$

Actually, we are interested in the average value of this quantity over the choice of the sample. Equation (12) tells us that the average minimum cost depends on the average of the *logarithm* of  $Z$ . This difficulty is usually circumvented by means of the so called ‘‘replica trick’’: one computes the average of  $Z^n$ , where  $n$  is an integer, and then the average of the logarithm is obtained by

$$\overline{\log Z} = \lim_{n \rightarrow 0} \frac{\partial \overline{Z^n}}{\partial n} , \quad (13)$$

thus assuming that  $Z^n$  can be analytically continued to real values of  $n$ . The overline stands for an average over different samples, i.e. over the probability distribution (10). This technique has a long history in the physics of spin glasses [9]: the proof that it leads to the correct solution has been obtained [10] recently.

The partition function (11) can be written more explicitly as

$$\begin{aligned} Z_\gamma[\{x_{i\tau}\}] = & \int_{-\infty}^{+\infty} dv \int_0^{+\infty} \prod_{\tau=1}^T du_\tau \int_{-\infty}^{+\infty} \prod_{i=1}^N dw_i \int_{-i\infty}^{+i\infty} d\lambda \exp \left[ \lambda \left( \sum_{i=1}^N w_i - N \right) \right] \times \\ & \times \int_0^{+\infty} \prod_{\tau=1}^T d\mu_\tau \int_{-i\infty}^{+i\infty} \prod_{\tau=1}^T d\hat{\mu}_\tau \exp \left[ \sum_{\tau=1}^T \hat{\mu}_\tau \left( u_\tau + v + \sum_{i=1}^N x_{i\tau} w_i - \mu_\tau \right) \right] \exp \left[ -\gamma(1-\beta)Tv - \gamma \sum_{\tau=1}^T u_\tau \right] \end{aligned} \quad (14)$$

where the constraints are imposed by means of the Lagrange multipliers  $\lambda, \mu, \hat{\mu}$ . In view of applying the trick in (13), we introduce  $n$  identical replicas of the system corresponding to the same history of returns  $\{x_{i\tau}\}$ , and write down  $Z_\gamma^n[\{x_{i\tau}\}]$ . After this, the average over the samples can be performed and allows one to introduce the *overlap* matrix

$$Q^{ab} = \frac{1}{N} \sum_{i=1}^N w_i^a w_i^b , \quad a, b = 1, \dots, n , \quad (15)$$

as well as its conjugate  $\hat{Q}^{ab}$  (the Lagrange multiplier imposing (15)). Here,  $a$  and  $b$  are replica indexes. After (several) Gaussian integrals, one is left with

$$\begin{aligned} \overline{Z_\gamma^n[\{x_{i\tau}\}]} \sim & \int_{-\infty}^{+\infty} \prod_{a=1}^n dv^a \int_{-\infty}^{+\infty} \prod_{a,b} dQ^{ab} \int_{-i\infty}^{+i\infty} \prod_{a,b} d\hat{Q}^{ab} \exp \left\{ N \sum_{a,b} Q^{ab} \hat{Q}^{ab} - N \sum_{a,b} \hat{Q}^{ab} - \gamma(1-\beta)T \sum_a v^a \right. \\ & \left. - Tn \log \gamma + T \log \hat{Z}_\gamma(\{v^a\}, \{Q^{ab}\}) - \frac{T}{2} \text{Tr} \log Q - \frac{N}{2} \text{Tr} \log \hat{Q} - \frac{nN}{2} \log 2 \right\} , \end{aligned} \quad (16)$$

where

$$\hat{Z}_\gamma(\{v^a\}, \{Q^{ab}\}) \equiv \int_{-\infty}^{+\infty} \prod_{a=1}^n dy^a \exp \left[ -\frac{1}{2} \sum_{a,b=1}^n (Q^{-1})^{ab} (y^a - v^a)(y^b - v^b) + \gamma \sum_{a=1}^n y^a \theta(-y^a) \right] . \quad (17)$$

We now write  $T = tN$  and work at fixed  $t$  while  $N \rightarrow \infty$ .

The most natural solution is obtained by realizing that all the replicas are identical. Given the linear character of the problem, the symmetric solution should be the correct one. The replica-symmetric solution corresponds to the *ansatz*

$$Q^{ab} = \begin{cases} q_1 & \text{if } a = b \\ q_0 & \text{if } a \neq b \end{cases} ; \quad \hat{Q}^{ab} = \begin{cases} \hat{q}_1 & \text{if } a = b \\ \hat{q}_0 & \text{if } a \neq b \end{cases} , \quad (18)$$

and  $v^a = v$  for any  $a$ . As we discuss in detail in appendix A, one can show that the optimal cost function, computed as from eq. (12) but with the average of the log, is the minimum of

$$\varepsilon(v, q_0, \Delta) = \frac{1}{2\Delta} + \Delta \left[ t(1-\beta)v - \frac{q_0}{2} + \frac{t}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} ds e^{-s^2} g(v + s\sqrt{2q_0}) \right] , \quad (19)$$

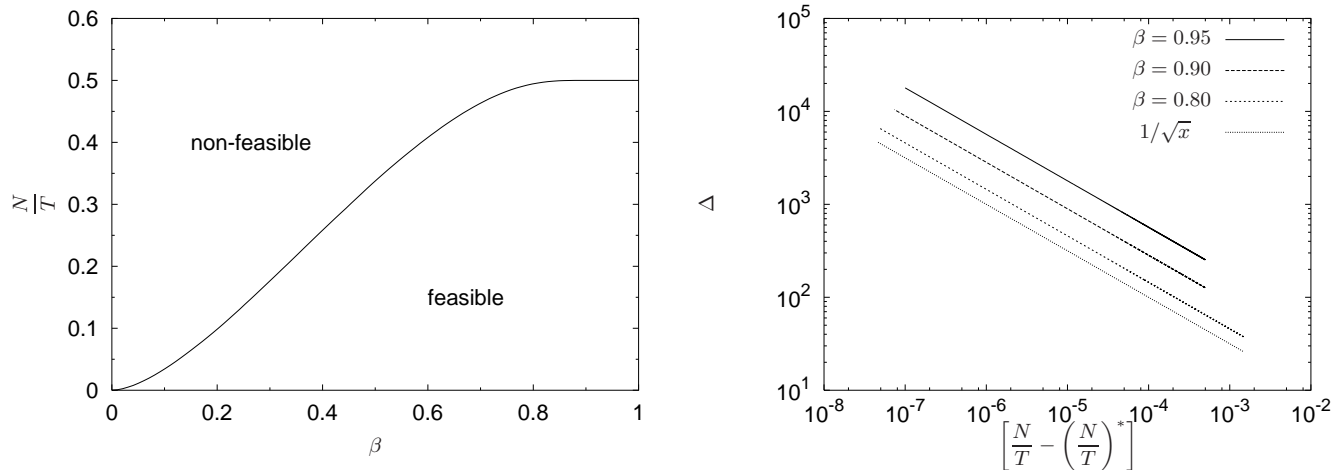


FIG. 2: Left panel: The phase diagram of the feasibility problem for expected shortfall. Right panel: The order parameter  $\Delta$  diverges with an exponent  $1/2$  as the transition line is approached. A curve of slope  $-1/2$  is also shown for comparison.

where  $\Delta \equiv \lim_{\gamma \rightarrow \infty} \gamma \Delta q$  and the function  $g(\cdot)$  is defined as

$$g(x) = \begin{cases} 0 & x \geq 0, \\ x^2 & -1 \leq x < 0, \\ -2x - 1 & x < -1. \end{cases} \quad (20)$$

Note that this function and its derivative are continuous. Moreover,  $v$  and  $q_0$  in (19) are solutions of the saddle point equations

$$1 - \beta + \frac{1}{2\sqrt{\pi}} \int ds e^{-s^2} g'(v + s\sqrt{2q_0}) = 0, \quad (21)$$

$$-1 + \frac{t}{\sqrt{2\pi q_0}} \int ds e^{-s^2} s g'(v + s\sqrt{2q_0}) = 0. \quad (22)$$

We require that the minimum of (19) occur at a finite value of  $\Delta$ . In order to understand this point, we recall the meaning of  $\Delta$  (see also (18)):

$$\Delta/\gamma \sim \Delta q = (q_1 - q_0) = \frac{1}{N} \sum_{i=1}^N (w_i^{(1)})^2 - \frac{1}{N} \sum_{i=1}^N w_i^{(1)} w_i^{(2)} \sim \overline{w^2} - \overline{w}^2, \quad (23)$$

where the superscripts (1) and (2) represent two generic replicas of the system. We then find that  $\Delta$  is proportional to the fluctuations in the distribution of the  $w$ 's. An infinite value of  $\Delta$  would then correspond to a portfolio which is infinitely short on some particular positions and, because of the global budget constraint (1), infinitely long on some other ones.

Given (19), the existence of a solution at finite  $\Delta$  translates into the following condition:

$$t(1 - \beta)v - \frac{q_0}{2} + \frac{t}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} ds e^{-s^2} g(v + s\sqrt{2q_0}) \geq 0, \quad (24)$$

which defines, along with eqs. (21) and (22), our phase diagram.

#### IV. THE PHASE DIAGRAM

We can now chart the feasibility map of the expected shortfall problem. We will use as control parameters  $N/T \equiv 1/t$  and  $\beta$ . The limiting case  $\beta \rightarrow 1$  can be worked out analytically and one can show that the critical value  $t^*$  is given by

$$\frac{1}{t^*} = \frac{1}{2} - \mathcal{O} \left[ (1 - \beta)^3 e^{-(4\pi(1-\beta)^2)^{-1}} \right]. \quad (25)$$

This limit corresponds to the over-pessimistic case of maximal loss, in which the single worst loss contribute to the risk measure. The optimization problem is the following:

$$\min_{\mathbf{w}} \left[ \max_{\tau \in \{1, \dots, T\}} \left( - \sum_i w_i x_{i\tau} \right) \right]. \quad (26)$$

A simple “geometric” argument by Kondor et al. [7] borrowed from a random geometry context [12] leads to the critical value  $1/t^* = 0.5$  in this extreme case. The idea is the following. According to eq. (26), one has to look for the minimum of a polytope made by a large number of planes, whose normal vectors (the  $x_{i\tau}$ ) are drawn from a symmetric distribution. The simplex is convex, but with some probability it can be unbounded from below and the optimization problem is ill defined. Increasing  $T$  means that the probability of this event decreases, because there are more planes and thus it is more likely that for large values of the  $w_i$  the max over  $t$  has a positive slope in the  $i$ -th direction. The exact law for this probability can be obtained by induction on  $N$  and  $T$  [7] and, as we said, it jumps in the thermodynamic limit from 1 to 0 at  $N/T = 0.5$ . Given that the corrections to this limit case are exponentially small (eq. (25)), the threshold 0.5 can be considered as a good approximation of the actual value for many cases of practical interest (i.e.  $\beta \gtrsim 0.9$ ).

For finite values of  $\beta$  we have solved numerically eqs. (21), (22) and (24) using the following procedure. We first solve the two equations (21) and (22), which always admit a solution for  $(v, q_0)$ . We then plot the l.h.s. of eq. (24) as a function of  $1/t$  for a fixed value of  $\beta$ . This function is positive at small  $1/t$  and becomes negative beyond a threshold  $1/t^*$ . By keeping track of  $1/t^*$  (numerically obtaining via linear interpolations) for each value of  $\beta$  we build up the phase diagram (Fig. 2, left). We show in the right panel of Fig. 2 the divergence of the order parameter  $\Delta$  versus  $1/t - 1/t^*$ . The critical exponent is found to be  $1/2$ :

$$\Delta \sim \left( \frac{1}{t} - \frac{1}{t^*(\beta)} \right)^{-1/2}. \quad (27)$$

, again in agreement with the scaling found in [7]. We have performed extensive numerical simulations in order to check the validity of our analytical findings. For a given realization of the time series, we solve the optimization problem (8) by standard linear programming [11]. We impose a large negative cutoff for the  $w$ 's, that is  $w_i > -W$ , and we say that a feasible solution exists if it stays finite for  $W \rightarrow \infty$ . We then repeat the procedure for a certain number of samples, and then average our final results (optimal cost, optimal  $v$ , and the variance of the  $w$ 's in the optimal portfolio) over those of them which produced a finite solution. In Fig. 3 we show how the probability of finding a finite solution depends on the size of the problem. Here, the probability is simply defined in terms of the frequency. We see that the convergence towards the expected 1-0 law is fairly slow, and a finite size scaling analysis is shown in the right panel. Without loss of generality, we can summarize the finite- $N$  numerical results by writing the probability of finding a finite solution as

$$p(N, T, \beta) = f \left[ \left( \frac{1}{t} - \frac{1}{t^*(\beta)} \right) \cdot N^{\alpha(\beta)} \right], \quad (28)$$

where  $f(x) \rightarrow 1$  if  $x \gg 1$  and  $f(x) \rightarrow 0$  if  $x \ll 1$ , and where  $\alpha(1) = 1/2$ .

In Fig. 4 (left panel) we plot, for a given value of  $\beta$ , the optimal cost found numerically for several values of the size  $N$  compared to the analytical prediction at infinite  $N$ . One can show that the cost vanishes as  $\Delta^{-1} \sim (1/t - 1/t^*)^{1/2}$ . The right panel of the same figure shows the behavior of the value of  $v$  which leads to the optimal cost versus  $N/T$ , for the same fixed value of  $\beta$ . Also in this case, the analytical ( $N \rightarrow \infty$  limit) is plotted for comparison. We note that this quantity was suggested [5] to be a good approximation of the VaR of the optimal portfolio: We find here that  $v_{\text{opt}}$  diverges at the critical threshold and becomes negative at an even smaller value of  $N/T$ .

## V. CONCLUSIONS

We have shown that the problem of optimizing a portfolio under the expected shortfall measure of risk by using empirical distributions of returns is not well defined when the ratio  $N/T$  of assets to data points is larger than a certain critical value. This value depends on the threshold  $\beta$  of the risk measure in a continuous way and this defines a phase diagram. The lower the value of  $\beta$ , the larger the length of the time series needed for the portfolio optimization. The analytical approach we have discussed in this paper allows us to have a clear understanding of this phase transition. The mathematical reason for the non-feasibility of the optimization problem is that, with a certain probability  $p(N, T, \beta)$ , the linear constraints in (9) define a simplex which is not bounded from below, thus leading to a solution which is not finite ( $\Delta q \rightarrow \infty$  in our language), in the same way as it happens in the extreme case  $\beta \rightarrow 1$

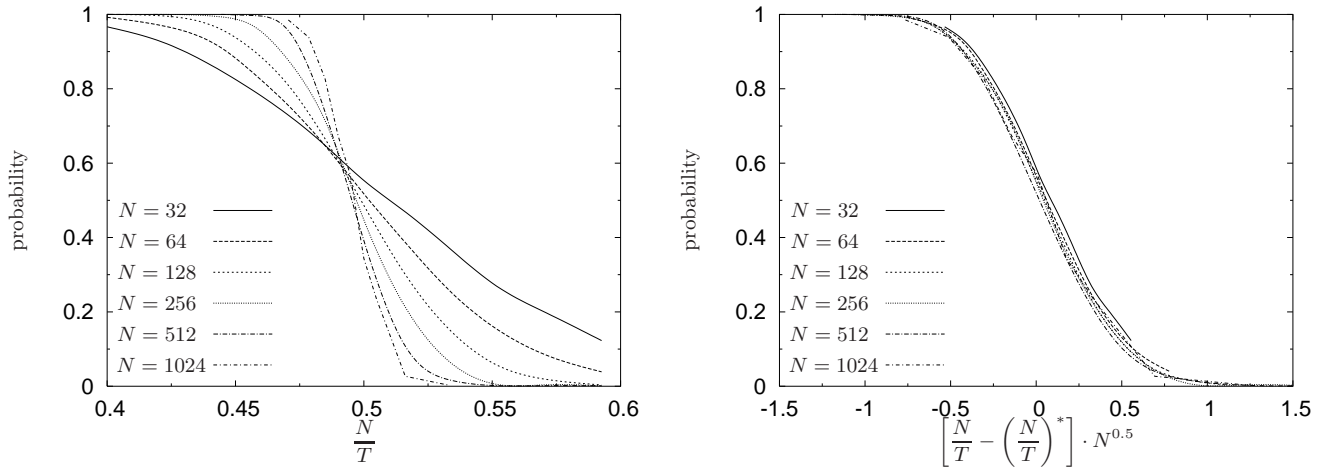


FIG. 3: Left: The probability of finding a finite solution as obtained from linear programming at increasing values of  $N$  and with  $\beta = 0.8$ . Right: Scaling plot of the same data. The critical value is set equal to the analytical one,  $N/T = 0.4945$  and the critical exponent is  $1/2$ , i.e. the one obtained in [7] for the limit case  $\beta \rightarrow 1$ . The data do not collapse perfectly, and better results can be obtained by slightly changing either the critical value or the exponent.

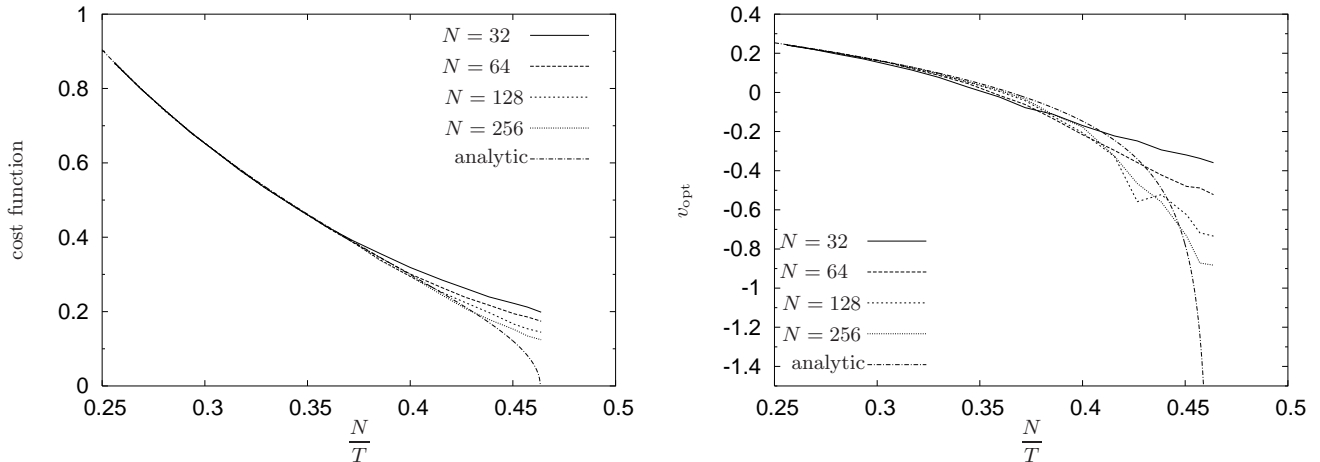


FIG. 4: Numerical results from linear programming and comparison with analytical predictions at large  $N$ . Left: The minimum cost of the optimization problem vs  $N/T$ , at increasing values of  $N$ . The thick line is the analytical solution (19). Here  $\beta = 0.7$ ,  $(N/T)^* \simeq 0.463$ . Right: The optimal value of  $v$  as found numerically for several values of  $N$  is compared to the analytical solution.

discussed in [7]. From a more physical point of view, it is reasonable that the feasibility of the problem depend on the number of data points we take from the time series with respect to the number of financial instruments of our portfolio. The probabilistic character of the time series is reflected in the probability  $p(N, T, \beta)$ . Interestingly, this probability becomes a threshold function at large  $N$  if  $N/T \equiv 1/t$  is finite, and its general form is given in (28).

These results have a practical relevance in portfolio optimization. The order parameter discussed in this paper is tightly related to the relative estimation error [6]. The fact that this order parameter has been found to diverge means that in some regions of the parameter space the estimation error blows up, which makes the task of portfolio optimization completely meaningless. The divergence of estimation error is not limited to the case of expected shortfall. As shown in [7], it happens in the case of variance and absolute deviation as well [14], but the noise sensitivity of expected shortfall turns out to be even greater than that of these more conventional risk measures.

There is nothing surprising about the fact that if there are no sufficient data, the estimation error is large and we cannot make a good decision. What is surprising is that there is a sharply defined threshold where the estimation error actually diverges.

For a given portfolio size, it is important to know that a minimum amount of data points is required in order to

perform an optimization based on empirical distributions. We also note that the divergence of the parameter  $\Delta$  at the phase transition, which is directly related to the fluctuations of the optimal portfolio, may play a dramatic role in practical cases. To stress this point, we can define a sort of ‘‘susceptibility’’ with respect to the data,

$$\chi_{ij}^\tau = \frac{\partial \langle w_j \rangle}{\partial x_{i\tau}}, \quad (29)$$

and one can show that this quantity diverges at the critical point, since  $\chi_{ij}^\tau \sim \Delta$ . A small change (or uncertainty) in  $x_{i\tau}$  becomes increasingly relevant as the transition is approached, and the portfolio optimization could then be very unstable even in the feasible region of the phase diagram. We stress that the susceptibility we have introduced might be considered as a measure of the effect of the noise on portfolio selection and is very reminiscent to the measure proposed in [6].

In order to present a clean, analytic picture, we have made several simplifying assumptions in this work. We have omitted the constraint on the returns, liquidity constraints, correlations between the assets, nonstationary effects, etc. Some of these can be systematically taken into account and we plan to return to these finer details in a subsequent work.

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## APPENDIX A: THE REPLICA SYMMETRIC SOLUTION

We show in this appendix how the minimum cost function corresponding to the replica-symmetric ansatz is obtained.

The ‘TrLog $Q$ ’ term in (16) is computed by realizing that the eigenvalues of such a symmetric matrix are  $(q_1 + (n - 1)q_0)$  (with multiplicity 1) and  $(q_1 - q_0)$  with multiplicity  $n - 1$ . Then,

$$\text{Tr} \log Q = \log \det Q = \log(q_1 + (n - 1)q_0) + (n - 1) \log(q_1 - q_0) = n \left( \log \Delta q + \frac{q_1}{\Delta q} \right) + \mathcal{O}(n^2), \quad (A1)$$

where  $\Delta q \equiv q_1 - q_0$ . The effective partition function in (17) depends on  $Q^{-1}$ , whose elements are:

$$(Q^{-1})^{ab} = \begin{cases} (\Delta q - q_0)/(\Delta q)^2 + \mathcal{O}(n) & \text{if } a = b \\ -q_0/(\Delta q)^2 + \mathcal{O}(n) & \text{if } a \neq b \end{cases} \quad (A2)$$

By introducing a Gaussian measure  $dP_{q_0}(s) \equiv \frac{ds}{\sqrt{2\pi q_0}} e^{-s^2/2q_0}$ , one can show that

$$\begin{aligned} \frac{1}{n} \log \hat{Z}(v, q_1, q_0) &= \frac{1}{n} \log \left\{ \int \prod_a dx_a e^{-\frac{1}{2\Delta q} \sum_a (x_a)^2 + \gamma \sum_a (x_a + v) \theta(-x_a - v)} \int dP_{q_0}(s) e^{\frac{s}{\Delta q} \sum_a x_a} \right\} \\ &= \frac{q_0}{2\Delta q} + \int dP_{q_0}(s) \log B_\gamma(s, v, \Delta q) + \mathcal{O}(n) \end{aligned} \quad (A3)$$

where we have defined

$$B_\gamma(s, v, \Delta q) \equiv \int dx \exp \left( -\frac{(x - s)^2}{2\Delta q} + \gamma(x + v) \theta(-x - v) \right). \quad (A4)$$

The exponential in (16) now reads  $\exp Nn[S(q_0, \Delta q, \hat{q}_0, \Delta \hat{q}) + \mathcal{O}(n)]$ , where

$$\begin{aligned} S(q_0, \Delta q, \hat{q}_0, \Delta \hat{q}) &= q_0 \Delta \hat{q} + \hat{q}_0 \Delta q + \Delta q \Delta \hat{q} - \Delta \hat{q} - \gamma t(1 - \beta)v - t \log \gamma + t \int dP_{q_0}(s) \log B_\gamma(s, v, \Delta q) \\ &\quad - \frac{t}{2} \log \Delta q - \frac{1}{2} \left( \log \Delta \hat{q} + \frac{\hat{q}_0}{\Delta \hat{q}} \right) - \frac{\log 2}{2}. \end{aligned} \quad (A5)$$

The saddle point equations for  $\hat{q}_0$  and  $\Delta \hat{q}$  allow then to simplify this expression. The free energy  $(-\gamma)f_\gamma = \lim_{n \rightarrow 0} \partial \overline{Z}_\gamma^n / \partial n$  is given by

$$-\gamma f_\gamma(v, q_0, \Delta q) = \frac{1}{2} - t \log \gamma + \frac{1 - t}{2} \log \Delta q + \frac{q_0 - 1}{2\Delta q} - \gamma t(1 - \beta)v + t \int dP_{q_0}(s) \log B_\gamma(s, v, \Delta q), \quad (A6)$$



where the actual values of  $v$ ,  $q_0$  and  $\Delta q$  are fixed by the saddle point equations

$$\frac{\partial f_\gamma}{\partial v} = \frac{\partial f_\gamma}{\partial q_0} = \frac{\partial f_\gamma}{\partial \Delta q} = 0 . \quad (\text{A7})$$

A close inspection of these saddle point equations allows one to perform the low temperature  $\gamma \rightarrow \infty$  limit by assuming that  $\Delta q = \Delta/\gamma$  while  $v$  and  $q_0$  do not depend on the temperature. In this limit one can show that

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\gamma} \log B_\gamma(s, v, \Delta/\gamma) = \begin{cases} s + v + \Delta/2 & s < -v - \Delta \\ -(v + s)^2/2\Delta & -v - \Delta \leq s < -v \\ 0 & s \geq -v \end{cases} \quad (\text{A8})$$

If we plug this expression into eq. (A6) and perform the large- $\gamma$  limit we get the minimum cost:

$$E = \lim_{\gamma \rightarrow \infty} f_\gamma = -\frac{q_0 - 1}{2\Delta} + t(1 - \beta)v - t \int_{-\infty}^{-\Delta} \frac{dx}{\sqrt{2\pi q_0}} e^{-\frac{(x-v)^2}{2q_0}} \left( x + \frac{\Delta}{2} \right) + \frac{t}{2\Delta} \int_{-\Delta}^0 \frac{dx}{\sqrt{2\pi q_0}} e^{-\frac{(x-v)^2}{2q_0}} x^2 . \quad (\text{A9})$$

We rescale  $x \rightarrow x\Delta$ ,  $v \rightarrow v\Delta$ , and  $q_0 \rightarrow q_0\Delta^2$ , and after some algebra we obtain eq. (19).

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